

Absolute and Relative Ambiguity Attitudes*

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Abstract

We represent preferences that exhibit absolute or relative attitudes towards ambiguity without assuming convexity of preferences. Our analysis is motivated by the recent experimental evidence by Baillon and Placido (2019) indicating that ambiguity becomes more tolerable as individuals are better off overall. Decreasing absolute ambiguity aversion is characterized by constant superadditive certainty equivalents and admits an act-dependent variational representation (Maccheroni et al., 2006). Decreasing relative ambiguity aversion relates to positive superhomogeneity and admits an act-dependent confidence preference representation (Chateauneuf and Faro, 2009). We apply our characterizations to retrieve a classic risk sharing result on the efficiency of trade and subjective beliefs of the individuals (Rigotti et al., 2008).

KEYWORDS: ambiguity aversion, absolute attitudes, relative attitudes.

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1 Introduction

The seminal work of Ellsberg (1961) originated a vast theoretical literature on ambiguity that successfully addressed several decision puzzles in the context of uncertainty. In parallel, a growing literature has employed these decision-making models to study a wide range of economic applications, e.g., risk sharing (Rigotti et al., 2008; Ghirardato and Siniscalchi, 2018a); moral hazard (Miao and Rivera, 2016); portfolio choice (Maccheroni et al., 2013). However, many popular ambiguity models fail to compel with the experimental evidence concerning how attitudes towards ambiguity change when individual welfare changes (Baillon and Placido, 2019). This paper attempts to fill this gap by providing general preference representations for absolute and relative ambiguity attitudes.

We depart from standard models of ambiguity aversion along two dimensions. First, we employ the notions of decreasing absolute ambiguity aversion and decreasing relative ambiguity aversion formalized by Xue (2020).¹ The former captures the idea that ambiguity becomes more tolerable when the individual is better off in absolute terms, and the latter that it becomes more tolerable when the relative size of the ambiguity the individual faces increases.² Despite their appeal, many ambiguity models assume independence notions which limit their ability to capture the whole spectrum of ambiguity attitudes. For instance, Choquet expected utility (Schmeidler, 1989) and (α -)maxmin expected utility (Gilboa and Schmeidler, 1989; Ghirardato et al., 2004) satisfy certainty independence, which implies both constant absolute and constant relative ambiguity aversion. Second, our analysis does not rely on any form of convexity of preferences. This allows us to avoid criticisms related to the Machina paradoxes (Machina, 2009; Baillon et al., 2011).

MBA preferences. Within the standard Anscombe and Aumann (1963) setup, we assume MBA preferences as primitives (Cerreia-Vioglio et al., 2011a). On top of completeness and transitivity, these preferences satisfy minimal requirements of rationality such as monotonicity, risk independence, and continuity conditions.³ Notably, we do not employ any unboundedness condition for our results.

¹Ghirardato and Siniscalchi proposed an axiom similar to decreasing absolute ambiguity aversion in their work “Compensated Absolute Ambiguity Attitudes,” a version of which was presented at RUD and D-TEA conferences in 2015. Our axiom of decreasing relative ambiguity aversion adapts the one of Xue (2020) by restricting to mixtures with respect to the worst consequence.

²As a byproduct, our analysis provides representation results for all the kinds of changing ambiguity attitudes, including constant, increasing absolute, and increasing relative ambiguity aversion.

³The acronym *MBA* refers to Monotone, Bernoulli, and Archimedean.

As a baseline for our analysis, we characterize MBA preferences (Lemma 1) in terms of certainty equivalent functionals representable as maxima of quasiconcave functions.⁴ Denote by \mathcal{F} the set of all acts $f : S \rightarrow X$ that map the set of states of the world S to set of consequences X . Lemma 1 shows that the certainty equivalent I associated with act $f \in \mathcal{F}$ can be written as

$$I(u(f)) = \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G \left(\int u(f) dp, p \right) \quad (1)$$

where u is an affine utility function, \mathcal{G} is a set of quasiconvex functions and monotone in the first argument, and $p \in \Delta(S)$ is a probability measure over the state space S .

We highlight two feasible interpretations for the representation (1). The first interpretation, advanced by Chandrasekher et al. (2022) and Xia (2020), rationalizes the decision problem as an intrapersonal game between two conflicting “selves”: Optimism, playing the best possible aggregator in \mathcal{G} , and Pessimism, selecting the worst possible belief in $\Delta(S)$. Alternatively, following Castagnoli et al. (2022), a second interpretation relates to the axiomatic treatment of risk mitigation by Drèze (1990). In this setting, decision makers are confident that choosing an action induces a probability measure over the state space without error. However, if we remove such confidence, we can relate the decision maker’s action as inducing an aggregator in \mathcal{G} and concerns for misspecification to the infimum among all possible probabilistic models in $\Delta(S)$.

Decreasing absolute ambiguity aversion. We enrich our baseline (Lemma 1) by considering preferences satisfying decreasing absolute ambiguity aversion. This axiom states that, if a mixture involving an act $f \in \mathcal{F}$ and a constant act $x \in X$, written $\alpha f + (1 - \alpha)x$ for $\alpha \in (0, 1)$, is preferred to $\alpha z + (1 - \alpha)x$, where $z \in X$, then this preference is preserved if we replace x in both acts with a better constant act $y \in X$.⁵

Theorem 1, which constitutes our main representation of decreasing absolute ambiguity aversion, formalizes the link between this property and constant superadditivity. Relating to the representation (1), Theorem 1 further imposes that each aggregator $G \in \mathcal{G}$ is constant superadditive in the first argument, that is,

$$G \left(\int u(f) + u(x) dp, p \right) \geq G \left(\int u(f) dp, p \right) + u(x)$$

⁴This result generalizes Theorem 4 in Chandrasekher et al. (2022).

⁵As customary, we identify constant acts with their consequence in X .

for every $f \in \mathcal{F}$, $x \in X$ with $u(x) \geq 0$, and $p \in \Delta(S)$. This result conforms to our intuition. Under decreasing absolute ambiguity aversion, $u(f) + u(x)$ is evaluated with a lower degree of ambiguity aversion than $u(f)$. Therefore, combining an uncertain prospect f with a positive, certain one, x , yields a higher utility than when the two prospects are considered separately, resulting in constant superadditivity.

The fact that Theorem 1 imposes constant superadditivity on each aggregator $G \in \mathcal{G}$ is insightful for our interpretation. When reading the problem as an intrapersonal game between two selves, it implies that the pessimistic self exhibits decreasing absolute ambiguity aversion irrespectively of the move of the optimistic self, that is, the choice of the aggregator. Instead, in a risk mitigation framework, it is as if the decision maker exhibits decreasing absolute ambiguity aversion independently of the chosen action.

Example 1 introduces a generalization of second-order expected utility (Grant et al., 2009; Neilson, 2010), which we term *second-order expected utility with risk mitigation*. Compared to the standard framework, our model features multiple probability distribution over the states rather than a single one, and the decision maker evaluates each act with the most favorable of these distributions. Following Drèze (1990), the decision maker behaves as if she can mitigate the uncertainty associated with the chosen act by taking an action that influences the distribution of the states. We show that if the function governing ambiguity attitudes satisfies decreasing absolute risk aversion (DARA), i.e., the Arrow-Pratt coefficient of absolute risk aversion is decreasing, then this model satisfies the representation of Theorem 1, displaying decreasing absolute ambiguity aversion.

Proposition 1 provides a second characterization of decreasing absolute ambiguity aversion yielding an act-dependent variational representation (Maccheroni et al., 2006). In particular, each act $f \in \mathcal{F}$ is evaluated as follows

$$I(u(f)) = \max_{c \in C_f} \min_{p \in \Delta(S)} \left\{ \int u(f) dp + c(p) \right\}$$

where the set C_f depends on the act f and collects lower semicontinuous and convex ambiguity costs, $c : \Delta(S) \rightarrow (-\infty, \infty]$. Decreasing absolute ambiguity aversion is captured by the fact that the set of the ambiguity costs enlarges as the decision maker evaluates more favorable acts, i.e., $u(g) \leq u(f)$ implies $C_g \subseteq C_f$, for $f, g \in \mathcal{F}$. In a dual-self perspective, as the prospect considered improves, the optimistic self is better off as she can choose an ambiguity cost from a larger set; in the risk mitigation interpretation, the cost related to misspecification concerns is attenuated.

Example 2 relates to the representation of Proposition 1 by introducing an act-dependent version of multiplier preferences (Hansen and Sargent, 2001). In this model, the decision maker can access an outside option, which yields a fixed utility level. Each act is associated with the set of distributions that justifies the choice of the act over the outside option; the set of ambiguity costs collects the relative entropies computed using these distributions. Since this set of ambiguity costs enlarges in the utility of the act, these preferences satisfy decreasing absolute ambiguity aversion.

Decreasing relative ambiguity aversion. This notion captures the idea that ambiguity aversion decreases as the proportion of the certainty part of the act decreases. To define it, we assume the existence of a worst consequence $x_* \in X$, i.e., $u(x_*) \leq u(y)$ for all $y \in X$, which we normalize to $u(x_*) = 0$. This axiom states that if, for all acts $f \in \mathcal{F}$, $y \in X$, and mixing weight $\alpha \in (0, 1)$, $\alpha f + (1 - \alpha)x_*$ is preferred to $\alpha y + (1 - \alpha)x_*$, then this preference is preserved if we replace α in both acts with a larger mixing weight $\beta \geq \alpha$.

Our axiom of decreasing relative ambiguity aversion is novel; it modifies the one proposed by Xue (2020) by restricting to mixtures involving the worst consequence only. To check the soundness of this axiom, we show that, in the context of the smooth ambiguity model (Klibanoff et al., 2005), and assuming ambiguity aversion, it is equivalent to requiring that the function governing ambiguity attitudes satisfies decreasing relative risk aversion (DRRA), i.e., the Arrow-Pratt coefficient of relative risk aversion is decreasing.⁶

Theorem 2, our main representation of decreasing relative ambiguity aversion, connects this property to positive superhomogeneity. Relating to the representation (1), Theorem 2 imposes that each aggregator $G \in \mathcal{G}$ is positively superhomogeneous in the first argument, namely,

$$G\left(\int u(\alpha f + (1 - \alpha)x_*)dp, p\right) \leq \alpha G\left(\int u(f)dp, p\right)$$

for every $\alpha \in (0, 1)$, $f \in \mathcal{F}$, and $p \in \Delta(S)$. When considering $\alpha f + (1 - \alpha)x_*$, as α increases, the proportion of the certainty part of the act decreases. Therefore, under decreasing relative ambiguity aversion, $\alpha f + (1 - \alpha)x_*$ is evaluated with a higher degree of ambiguity aversion than f , resulting in positive superhomogeneity.

As for absolute attitudes, Theorem 2 implies that positive superhomogeneity holds for

⁶See Remark 2 for further details. The analog result for decreasing absolute ambiguity aversion in the context of the smooth model is shown by Xue (2020) in Proposition 7.

all aggregators. Interpreting the problem as an intrapersonal game between two selves, the pessimistic self exhibits decreasing relative ambiguity aversion irrespectively of the move of the optimistic self; in a risk mitigation framework, it is as if the decision maker exhibits decreasing relative ambiguity aversion independently of the chosen action.

Analogously to Example 1, Example 3 connects second-order expected utility with risk mitigation to the representation of Theorem 2. In particular, we show that if the function governing ambiguity attitudes satisfies DRRA then this model can be represented as in Theorem 2, hence displaying decreasing relative ambiguity aversion.

Proposition 2 provides an alternative representation of decreasing relative ambiguity aversion, characterizing this property in terms of an act-dependent confidence model (Chateauneuf and Faro, 2009). In particular, each act $f \in \mathcal{F}$ is evaluated as follows

$$I(u(f)) = \max_{d \in D_f} \min_{p \in \Delta(S)} \frac{\int_S u(f) dp}{d(p)}$$

where the set D_f depends on the act f and collects upper semicontinuous and quasi-concave confidence functions, $d : \Delta(S) \rightarrow [0, \infty]$. As for absolute attitudes, decreasing relative ambiguity aversion is captured by the set of confidence functions enlarging as the decision maker evaluates more favorable acts, i.e., $u(g) \leq u(f)$ implies $D_g \subseteq D_f$, for $f, g \in \mathcal{F}$. This feature implies positive superhomogeneity conforming to Theorem 2.

Example 4 relates to the representation of Proposition 2 by introducing an act-dependent model of entropic-confidence preferences. As in Example 2, the decision maker can access an outside option and associate each act with the set of distributions that justifies the act over the outside option; the confidence functions are the relative entropies computed using these distributions. Since this set of functions enlarges in the utility of the act, these preferences satisfy decreasing relative ambiguity aversion.

Risk sharing application. We apply our characterizations of absolute and relative ambiguity attitudes to investigate whether, in financial markets, it is efficient for agents displaying general ambiguity preferences to engage in speculative betting. In an exchange economy with a single consumption good and no aggregate uncertainty, subjective expected utility, risk averse agents introduce individual uncertainty in the final allocation, that is, they engage in betting, if and only if their beliefs differ (Milgrom and Stokey, 1982). This result has been extended, first by Billot et al. (2000) in the context of maxmin expected utility, and later by Rigotti et al. (2008) for general convex preferences, by showing that

agents will bet if and only if they do not share *any* beliefs, i.e., their sets of subjective beliefs, properly defined, do not intersect. Finally, Ghirardato and Siniscalchi (2018a) generalize this result to non-convex preferences, proposing a condition called *strict pseudoconcavity at certainty* which, loosely speaking, requires the indifference curves at a consumption point to lie strictly above the tangent line.

Proposition 3 connects these risk sharing results to our analysis of changing ambiguity attitudes. In particular, it shows that our characterization of decreasing absolute ambiguity aversion joint with the one of increasing relative ambiguity aversion,⁷ plus some regularity requirements, implies strict pseudoconcavity at certainty, providing a novel behavioral interpretation for this condition. The inefficiency of betting is then implied by Ghirardato and Siniscalchi (2018a).

In contrast, in Example 5, we argue that decreasing relative ambiguity aversion might prompt agents to bet even if they share the same beliefs. The key intuition is that Theorem 2 links decreasing relative ambiguity aversion to a representation that is “convex at 0,” leading agents to prefer uncertainty over full insurance.

Related literature. We contribute directly to the decision-theoretic literature investigating changing ambiguity attitudes resulting from utility shifts. Grant and Polak (2013) focus on the case of constant absolute ambiguity aversion and show that this property admits a mean dispersion representation. We instead consider non-constant absolute and relative attitudes. To this end, we borrow the axioms of decreasing absolute ambiguity aversion and decreasing relative ambiguity aversion, the latter with some modifications, from the analysis of Xue (2020).⁸ Like us, Xue (2020) studies changing ambiguity attitudes resulting from utility shifts, but focuses only on the case of ambiguity averse preferences.

A different approach is pioneered by Cherbonnier and Gollier (2015) and Cerreia-Vioglio et al. (2022), which consider wealth effects, instead of utility shifts. In this framework, Cerreia-Vioglio et al. (2022) show that wealth-classifiable preferences, that is, ambiguity preferences that are either increasing, constant, or decreasing to wealth changes in absolute or relative terms, must necessarily display constant absolute risk aversion. Therefore, one can view studies on utility shifts as a way to overcome this restrictive assumption on risk preferences. Employing a different notion of comparative ambiguity attitudes, Wang (2019) studies wealth effects without assumptions on risk preferences.

⁷The representation of increasing relative ambiguity aversion follows the one of Theorem 2 for decreasing relative ambiguity aversion by replacing positively superhomogeneity with positively subhomogeneity.

⁸See also Chambers et al. (2014) for the related notion of absolute uncertainty attitudes.

Our work is motivated by the recent experimental evidence of Baillon and Placido (2019). At the aggregate level, their findings support decreasing absolute and decreasing relative ambiguity aversion. Roughly, 40% of the subjects of their experiments display constant absolute ambiguity aversion and 40% display decreasing absolute ambiguity aversion; almost half of the subjects satisfy decreasing relative ambiguity aversion. Furthermore, their analysis suggests that constant absolute ambiguity aversion would not make accurate predictions for most subjects unless we accept errors of up to 10%. This evidence highlights the inadequacy of popular decision-making models,⁹ to capture experimentally corroborated ambiguity attitudes. See section 3.1 for a discussion.

A growing literature studies the consequences of decreasing absolute ambiguity aversion in economic applications. Some of these applications interpret this property as a form of ambiguity prudence and use it to explain precautionary savings (Berger, 2014; Osaki and Schlesinger, 2014) and self-protective behavior (Berger, 2016). Other applications study market selection. Guerdjikova and Sciubba (2015) show that decision makers displaying decreasing absolute ambiguity aversion survive in markets populated by expected utility agents. In a framework that allows for a broad class of recursive preferences, Beker and Chen (2023) show that under decreasing absolute ambiguity aversion, every full-support belief can survive if there is sufficiently high uncertainty.

We borrow technical insights from the literature on risk measures. In particular, some of our results are inspired by Han et al. (2022) that provide representations as minima of constant subadditive and quasiconvex functions. Similar results for positively superhomogeneous functionals appear in: Castagnoli et al. (2022), which characterize star-shaped monetary, i.e., constant additive, risk measures;¹⁰ Laeven et al. (2023), which provide a representation of star-shaped functionals in terms of convex and lower semicontinuous functionals. Our results distinguish from these since, without assuming unboundedness, we have to employ extension techniques and envelope continuity results.

2 Mathematical preliminaries

We consider the Anscombe and Aumann (1963) setup composed of a nonempty set S of *states of the world*, endowed with an algebra Σ of subsets of S called *events*, and a

⁹These models include Choquet (Schmeidler, 1989), maxmin (Gilboa and Schmeidler, 1989), α -maxmin (Ghirardato et al., 2004), dual-self (Chandrasekher et al., 2022), variational (Maccheroni et al., 2006), vector expected utility (Siniscalchi, 2009), and confidence preferences (Chateauneuf and Faro, 2009).

¹⁰Cerreia-Vioglio et al. (2024) retrieve analogous results with additional subdifferential properties.

nonempty convex set X of *consequences*. Denote by $\Delta(S)$ the set of finitely additive probability measures over S .

The decision maker has preferences over the set \mathcal{F} of all (*simple*) acts, i.e., Σ -measurable functions $f : S \rightarrow X$ such that $f(S)$ is a finite set. For all $x \in X$, we identify $x \in \mathcal{F}$ as the constant act equals to x , and, as a result, X as a subset of \mathcal{F} . For all $f, g \in \mathcal{F}$, and $\alpha \in [0, 1]$, relying on the linear structure of X , we define convex combinations of acts as

$$(\alpha f + (1 - \alpha)g) : s \mapsto \alpha f(s) + (1 - \alpha)g(s) \in X.$$

We denote by \succsim a binary relation over \mathcal{F} , and by \succ and \sim its asymmetric and symmetric parts, respectively. For all $f \in \mathcal{F}$, denote by $x_f \in X$ a *certainty equivalent* of f , i.e., $x_f \sim f$. A function $V : \mathcal{F} \rightarrow \mathbb{R}$ is a *utility representation* for \succsim if, for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff V(f) \geq V(g).$$

Whenever the decision maker's preferences over consequences admit a utility representation, comparisons between acts can be expressed in utility levels. To formalize this, we introduce $B_0(\Sigma, K)$, the set of Σ -measurable real-valued bounded simple functions which images are included in K , interpreted as the set of utility levels, for some $K \subseteq \mathbb{R}$. To ease notation, let $B_0(\Sigma) = B_0(\Sigma, \mathbb{R})$. Endow these sets with the supnorm $\|\cdot\|_\infty$.

As customary in decision theory under ambiguity, we study the properties of “certainty equivalent functionals.” To this end, we introduce some properties discussed in the upcoming sections. Fix a convex set $K \subseteq \mathbb{R}$. A functional $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ is *normalized* if $I(k) = k$ for all $k \in K$; *monotone* if $\varphi \geq \psi$ implies $I(\varphi) \geq I(\psi)$, for all $\varphi, \psi \in B_0(\Sigma, K)$, and *strictly monotone* if in addition $\varphi \neq \psi$ implies $I(\varphi) > I(\psi)$; *quasiconcave* if for all $\varphi, \psi \in B_0(\Sigma, K)$, and $\alpha \in [0, 1]$, we have

$$I(\alpha\varphi + (1 - \alpha)\psi) \geq \min\{I(\varphi), I(\psi)\};$$

quasiconvex if $-I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ is quasiconcave.

The notions of constant super and subadditivity are particularly relevant for our analysis of absolute ambiguity attitudes. Specifically, I is *constant superadditive* if, for all $\varphi \in B_0(\Sigma, K)$, and $k \geq 0$, such that $\varphi + k \in B_0(\Sigma, K)$, we have

$$I(\varphi + k) \geq I(\varphi) + k. \tag{2}$$

Analogously, I is *constant subadditive* if equation (2) holds with the reverted inequality; *constant additive* if it is both constant superadditive and constant subadditive. Constant superadditivity and subadditivity are *strict* when they hold with the strict inequality for every $\varphi \neq 0$ and $k > 0$.

For relative ambiguity attitudes, we introduce the concepts of positive super and subhomogeneity. In particular, I is *positively superhomogeneous* if

$$I(\gamma\varphi) \leq \gamma I(\varphi) \quad (3)$$

for all $\gamma \in (0, 1)$, and $\varphi \in B_0(\Sigma, K)$ with $\gamma\varphi \in B_0(\Sigma, K)$; *positive subhomogeneity* is defined by reverting the inequality (3); *positive homogeneity* holds if I is both positively super and subhomogeneous. Positive superhomogeneity and subhomogeneity are *strict* when they hold with the strict inequality for every $\varphi \neq 0$.

Our representations satisfy additional regularity requirements. A family Υ of functions $H: B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ is *regular* if: (i) $\varphi \mapsto \max_{H \in \Upsilon} H(\varphi)$ is well-defined and continuous, (ii) if K is lower open,¹¹ then each element of Υ is lower semicontinuous, and (iii) if K is either upper open or a bounded interval, then each element of Υ is upper semicontinuous. Moreover, a family \mathcal{G} of functions $G: \mathbb{R} \times \Delta(S) \rightarrow \bar{\mathbb{R}}$ is *linearly continuous* if

$$\varphi \mapsto \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int \varphi dp, p\right)$$

is continuous.

3 Monotone, Bernoulli, Archimedean preferences

We now introduce the axioms, maintained throughout our analysis, characterizing MBA preferences. Let a binary relation \succsim on \mathcal{F} represent the decision maker's preferences.

AXIOM (weak order). \succsim is *complete and transitive*.

AXIOM (risk independence). If $x, y, z \in X$, and $\alpha \in (0, 1)$,

$$x \sim y \implies \alpha x + (1 - \alpha)z \sim \alpha y + (1 - \alpha)z.$$

¹¹A subset K of \mathbb{R} is *lower open* (resp. *upper open*) if, for all $k \in K$, there exists $\varepsilon > 0$ such that $[k - \varepsilon, k] \subseteq K$ (resp. $[k, k + \varepsilon] \subseteq K$).

AXIOM (archimedean continuity). *If $f, g, h \in \mathcal{F}$ and $f \succ g \succ h$, then there exist $\alpha, \beta \in [0, 1]$ such that*

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

AXIOM (monotonicity). *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for each $s \in S$, then $f \succsim g$.*

A binary relation \succsim on \mathcal{F} is an *MBA preference*, or simply *MBA*, if it satisfies weak order, risk independence, archimedean continuity, and monotonicity.

Lemma 1 below provides a representation of MBA preferences which constitutes the starting point of our analysis and connects existing results in the literature. Cerreia-Vioglio et al. (2011a) show that a binary relation \succsim is MBA if and only if there exist: (i) an affine function $u : X \rightarrow \mathbb{R}$, and (ii) a monotone, normalized, and continuous functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that $I \circ u$ represents \succsim . We sharpen their representation showing that the certainty equivalent I can be taken as the maximum over a set of monotone and quasiconcave functions. A similar result appears also in Chandrasekher et al. (2022) (Theorem 4), but limited to finite state spaces. We extend their result to arbitrary state spaces, and establish additional regularity conditions.

LEMMA 1. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) *\succsim is an MBA preference relation.*
- (ii) *There exist an affine function $u : X \rightarrow \mathbb{R}$, and a regular set Ψ of monotone, quasiconcave functions $H : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that*

$$f \succsim g \iff \max_{H \in \Psi} H(u(f)) \geq \max_{H \in \Psi} H(u(g)) \quad (4)$$

for all $f, g \in \mathcal{F}$ and $\max_{H \in \Psi} H(u(x)) = u(x)$ for all $x \in X$.

We highlight two feasible interpretations for our representation of MBA preferences. Using quasiconcave duality results by Cerreia-Vioglio et al. (2011b), the utility representation in (4) can be written as

$$V(f) = \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int u(f) dp, p\right) \quad (5)$$

where \mathcal{G} is a linearly continuous family of quasiconvex functions monotone in the first argument. The first interpretation, by Chandrasekher et al. (2022) and Xia (2020), rationalizes the decision problem as an intrapersonal game between two conflicting “selves”:

Optimism, playing the best possible aggregator in \mathcal{G} , and Pessimism, selecting the worst possible belief in $\Delta(S)$. Alternatively, following Castagnoli et al. (2022), a second interpretation relates to the axiomatic treatment of risk mitigation by Drèze (1990). In Drèze, decision makers are confident that choosing an action induces a probability measure over the state space without error. By allowing for less confident decision makers, we interpret the decision maker's action as inducing an aggregator in \mathcal{G} and concerns for misspecification as the infimum among all possible probabilistic models in $\Delta(S)$.

In the following sections, we enrich our representation of MBA preferences by studying the role of absolute and relative ambiguity attitudes. To this end, we first discuss how other ambiguity models relates to these properties.

3.1 Independence notions and ambiguity attitudes

Popular models based on MBA preferences are neutral towards absolute and/or relative changes in utility levels. This neutrality is implied by distinct notions of independence, stronger than risk independence. To discuss this aspect further, we formally introduce constant absolute ambiguity aversion (Grant and Polak, 2013), and constant relative ambiguity aversion.

AXIOM (constant absolute ambiguity aversion). *For all $f \in \mathcal{F}$, $x, y, z \in X$, and $\alpha \in (0, 1)$,*

$$\alpha f + (1 - \alpha)x \succsim \alpha z + (1 - \alpha)x \implies \alpha f + (1 - \alpha)y \succsim \alpha z + (1 - \alpha)y.$$

To define relative ambiguity attitudes, we assume that the binary relation \succsim on \mathcal{F} admits a *worst consequence*. That is, there exists $x_* \in X$ such that $y \succsim x_*$ for all $y \in X$.

AXIOM (constant relative ambiguity aversion). *For all $f \in \mathcal{F}$, $x, y \in X$, and $\alpha, \beta \in (0, 1)$,*

$$\alpha f + (1 - \alpha)x_* \succsim \alpha y + (1 - \alpha)x_* \implies \beta f + (1 - \beta)x_* \succsim \beta y + (1 - \beta)x_*.$$

Both these axioms impose a form of “independence.” Constant absolute ambiguity aversion implies independence in absolute changes in utility levels, meaning the decision maker ambiguity aversion is invariant towards absolute utility shifts. To see this, notice that the implication in the axiom does not depend on the preference ranking of the constant acts x and y . Similarly, constant relative ambiguity aversion requires independence towards relative changes in utility levels. This follows since the implication in the axiom

does not depend on whether the weight α is larger or smaller than the weight β .

Below, we summarize the independence notions of popular MBA preference models, highlighting their relation to constant absolute and constant relative ambiguity attitudes.

- *Choquet (Schmeidler, 1989), maxmin (Gilboa and Schmeidler, 1989), α -maxmin (Ghirardato et al., 2004), and dual-self maxmin (Chandrasekher et al., 2022).*

All these models imply *certainty independence*, which requires that preferences are independent of mixtures with constant acts. Formally, a binary relation \succsim on \mathcal{F} satisfies certainty independence if, for all $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in [0, 1]$,

$$f \succsim g \iff \alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x.$$

It can be seen how certainty independence implies both constant absolute ambiguity aversion and constant relative ambiguity aversion.

- *Variational (Maccheroni et al., 2006), and vector expected utility (Siniscalchi, 2009).*

These preferences satisfy a weaker version of certainty independence, named *weak certainty independence*. A binary relation \succsim on \mathcal{F} satisfies weak certainty independence if, for all $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0, 1)$,

$$\alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x \implies \alpha f + (1 - \alpha)y \succsim \alpha g + (1 - \alpha)y.$$

It is immediate to verify that this notion implies constant absolute ambiguity aversion. Moreover, it allows for dependence of ambiguity aversion on relative utility shifts. Indeed, it requires that preferences are independent to mixtures with constant acts while keeping the relative mixing weights, α and $1 - \alpha$, constant.¹²

- *Confidence preferences (Chateauneuf and Faro, 2009).*

This model satisfies *worst independence*, imposing independence to mixtures involving the worst consequence. Formally, a binary relation \succsim on \mathcal{F} satisfies worst independence if, for all $f, g \in \mathcal{F}$, and $\alpha \in (0, 1)$,

$$f \sim g \implies \alpha f + (1 - \alpha)x^* \sim \alpha g + (1 - \alpha)x^*.$$

¹²Variational preferences are represented by a concave and normalized certainty equivalent and hence satisfy increasing relative ambiguity aversion. This contrasts with the maxmin model, where the certainty independence axiom implies positive homogeneity and, in turn, constant relative ambiguity aversion.

For MBA preferences, worst independence is equivalent to constant relative ambiguity aversion, as both lead to positive homogeneous certainty equivalents. Furthermore, specularly to weak certainty independence, this axiom allows for the dependence of ambiguity aversion on absolute utility shifts by requiring independence only with respect to relative mixing weights, keeping fixed the worst consequence on all mixtures.¹³

4 Absolute ambiguity attitudes

In this section, we characterize preferences that display less aversion to ambiguity as the decision maker's baseline utility increases. To this end, we employ Xue's (2020) axiom of decreasing absolute ambiguity aversion.

AXIOM (decreasing absolute ambiguity aversion). *For all $f \in \mathcal{F}$, $x, y, z \in X$, and $\alpha \in (0, 1)$, if $y \succsim x$, then*

$$\alpha f + (1 - \alpha)x \succsim \alpha z + (1 - \alpha)x \implies \alpha f + (1 - \alpha)y \succsim \alpha z + (1 - \alpha)y.$$

Contrarily to the constant absolute case, decreasing absolute ambiguity aversion allows the decision maker's ambiguity aversion to depend on absolute changes in utility levels. In particular, the axiom says that if an ambiguous act $\alpha f + (1 - \alpha)x$ is preferred over a constant act $\alpha z + (1 - \alpha)x$, then such a ranking is preserved if the certainty part improves from x to y on both sides. This axiom captures the idea that ambiguity becomes more tolerable when the decision maker is better off in absolute terms.

The following theorem provides our main representation of decreasing absolute ambiguity aversion. It connects this property to the representation (5) of Lemma 1 by imposing constant superadditivity of all aggregators.

THEOREM 1. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) *\succsim is MBA and exhibits decreasing absolute ambiguity aversion.*
- (ii) *There exist an affine function $u : X \rightarrow \mathbb{R}$ and a linearly continuous family \mathcal{G} of monotone, constant superadditive in the first argument, and quasiconvex functions*

¹³Since such preferences are represented by a superlinear and normalized functional, they exhibit decreasing absolute ambiguity aversion.

$G : \mathbb{R} \times \Delta(S) \rightarrow \bar{\mathbb{R}}$ such that

$$f \succsim g \iff \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G \left(\int u(f) dp, p \right) \geq \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G \left(\int u(g) dp, p \right)$$

for all $f, g \in \mathcal{F}$ and $\max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G \left(\int u(x) dp, p \right) = u(x)$ for all $x \in X$.

The two interpretations provided for Lemma 1 apply also for Theorem 1. The main distinction being the fact that now the representation captures decreasing absolute ambiguity aversion through the constant superadditivity of each aggregator $G(\cdot, p)$ for all $p \in \Delta(S)$. Interpreting again the decision problem as an intrapersonal game between two selves (Chandrasekher et al., 2022), this means that the pessimistic self exhibits decreasing absolute ambiguity aversion irrespectively of the move of the optimistic self, that is, the choice of the aggregator. Instead, in a risk mitigation framework (Drèze, 1990), it is as if the decision maker exhibits decreasing absolute ambiguity aversion independently of the chosen action. Both interpretations have a common ground: decreasing absolute ambiguity aversion ascribes to the aggregators only.

The following example introduces a generalization of second-order expected utility (Grant et al., 2009; Neilson, 2010), which allows for multiple probability distributions rather than a single one. We show that whenever the function governing ambiguity attitudes satisfies decreasing absolute risk aversion (DARA), i.e., the Arrow-Pratt coefficient of absolute risk aversion is decreasing, then the model can be written as the representation of Theorem 1, hence displaying decreasing absolute ambiguity aversion.

EXAMPLE 1 (second-order expected utility with risk mitigation; DARA). We model a decision maker with a finite set of probabilities $Q \subseteq \Delta^\sigma(S)$,¹⁴ where (S, Σ) is a measurable space, an affine utility over consequences satisfying $u(X) = [0, \infty)$, and ambiguity attitudes represented by a continuous, strictly increasing, and concave function $\phi : [0, \infty) \rightarrow \mathbb{R}$. For every act $f \in \mathcal{F}$, the second-order expected utility given $q \in Q$ is defined as

$$I_q(u(f)) = \phi^{-1} \left(\int \phi(u(f)) dq \right).$$

To aggregate models in Q , the decision maker uses the following criterion

$$V(f) = \max_{q \in Q} I_q(u(f)).$$

¹⁴We denote by $\Delta^\sigma(S)$ the set of countably additive probability measures over S .

Clearly, if Q is a singleton, these preferences collapse to second-order expected utility. In general, inspired by Drèze (1990), we interpret these preferences as reflecting a two-step procedure: first, the choice of an ambiguous alternative $f \in \mathcal{F}$, and second the choice of an action, $q \in Q$, which partially controls the probability over the states to mitigate the uncertainty involving f . For this reason, we refer to these preferences as *second-order expected utility with risk mitigation*.

Through the properties of our representation of Theorem 1, we investigate which assumptions on ϕ imply decreasing absolute ambiguity aversion. As each I_q is continuous and Q is finite, V is also continuous; as each I_q is monotone and normalized, V is monotone and normalized. By Lemma 11 and 12 in Appendix B.3, we have that, if ϕ is twice differentiable and satisfies DARA, i.e., $t \mapsto -\phi''(t)/\phi'(t)$ is decreasing, then each I_q is constant superadditive.¹⁵ As a consequence, such preferences exhibit decreasing absolute ambiguity aversion and admit a representation as in Theorem 1, with

$$G_q(t, p) = \sup \left\{ I_q(u(f)) : f \in \mathcal{F} \text{ and } \int u(f) dp \leq t \right\}$$

for all $(t, p) \in \mathbb{R} \times \Delta(S)$. The constant superadditivity of each I_q implies that each $G_q(\cdot, p)$ is constant superadditive as well. \blacktriangleleft

The following proposition relates decreasing absolute ambiguity aversion to an act-dependent variational representation.¹⁶ In their standard formulation (Maccheroni et al., 2006), these preferences are characterized by a single ambiguity cost $c : \Delta(S) \rightarrow [0, \infty]$ capturing the level of ambiguity aversion. Due to changing ambiguity attitudes and the absence of convexity of preferences, the set of ambiguity costs varies with the acts.

PROPOSITION 1. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) *\succsim is MBA and exhibits decreasing absolute ambiguity aversion.*
- (ii) *There exist an affine function $u : X \rightarrow \mathbb{R}$, and, for all $f \in \mathcal{F}$, a family C_f of lower semicontinuous and convex functions $c : \Delta(S) \rightarrow (-\infty, \infty]$ such that*

$$f \succsim g \iff \max_{c \in C_f} \min_{p \in \Delta(S)} \left\{ \int_S u(f) dp + c(p) \right\} \geq \max_{c \in C_g} \min_{p \in \Delta(S)} \left\{ \int_S u(g) dp + c(p) \right\},$$

where $\max_{c \in C_x} \min_{p \in \Delta(S)} c(p) = 0$ for all $x \in X$, and $C_g \subseteq C_f$ for all $f, g \in \mathcal{F}$ with $u(f) \geq u(g)$.

¹⁵The function $\phi : t \mapsto \sqrt{t}$ satisfies all the listed hypotheses.

¹⁶Proposition 1 extends Proposition A.3 in Han et al. (2022) by removing unboundedness conditions.

This result generalizes the dual-self variational representation in Chandrasekher et al. (2022) (Theorem 3), which differs from ours as it is act-independent, a consequence of constant absolute ambiguity aversion. Instead, due to decreasing absolute ambiguity aversion, the set of ambiguity costs varies with the acts and enlarges as the utility levels increase. In the intrapersonal game interpretation, this reflects in the optimistic self being allowed to select a more favorable cost; while in the risk mitigation interpretation in an attenuation of the costs related to misspecification concerns. In general, it highlights how the higher the utility levels, the lower the ambiguity costs the decision maker faces.

In Proposition 1 point (ii), each cost function belonging to C_f takes values in $(-\infty, \infty]$. This is done in light of the following remark, which provides a nice construction of these costs. Alternatively, each cost function could be taken with values in $[0, \infty]$ and the construction would rely, following Maccheroni et al. (2006) and Cerreia-Vioglio et al. (2014), on the more standard Fenchel-Moreau representation.

REMARK 1. As observed by Han et al. (2022) (Proposition A.3), the family of ambiguity costs in the act-dependent variational representation can be written in a specific form. In particular, by inspecting the proof of their Proposition A.3, it follows that

$$I(u(f)) = \max_{g: u(f) \geq u(g)} \min_{p \in \Delta(S)} \left\{ \int_S u(f) dp + u(x_g) - \int_S u(g) dp \right\} \quad (6)$$

for all $f \in \mathcal{F}$. This representation, which is equivalent to the one of Proposition 1, has the advantage of making explicit the following structure of ambiguity costs

$$C_f = \left\{ p \mapsto u(x_g) - \int_S u(g) dp : g \in \mathcal{F}, \forall s \in S, f(s) \succeq g(s) \right\}.$$

$$c_g^i : p \mapsto u(x_g^i) - \int_S u(g) dp$$

Notice that, even considering the more explicit form of this representation, in the absence of uniqueness-related results, many other families of ambiguity indexes may exist. ◀

The following example relates to the representation of decreasing absolute ambiguity aversion of Proposition 1 by considering an act-dependent version of multiplier preferences (Hansen and Sargent, 2001).

EXAMPLE 2 (multiplier preferences with outside option). We model a decision maker that can mitigate the uncertainty induced by any act $f \in \mathcal{F}$ by exerting some effort that

affects the probability distribution over a finite set of states S , where $\Sigma = 2^S$. In particular, exerting effort to mitigate the uncertainty of f is valuable if it can induce a model, from the finite set $Q \subseteq \Delta(S)$, which makes f preferred over an outside option yielding a utility level equal to $\theta \in [0, \infty)$. If no model in Q justifies the choice of f over the outside option, then exerting effort is not valuable, and the decision maker evaluates f using as benchmark q_u , the uniform distribution over S . Formally, each act f is associated a set of probability distributions

$$C_f = \left\{ q \in Q : \int u(f) dq \geq \theta \right\} \cup \{q_u\}.$$

Notice that, for $f, g \in \mathcal{F}$, $u(f) \leq u(g)$ implies $C_f \subseteq C_g$.

The decision maker is concerned with model misspecification and displays multiplier preferences

$$V(f) = \max_{q \in C_f} \min_{p \in \Delta(S)} \left\{ \int u(f) dp + \lambda R(p \| q) \right\} \quad (7)$$

where $\lambda > 0$, and $R(\cdot \| \cdot)$ denotes the relative entropy which can be defined, for every $p, q \in \Delta(S)$, as

$$R(p \| q) = \int \log\left(\frac{dp}{dq}\right) dp$$

if $p \ll q$; $+\infty$ otherwise. These preferences satisfy decreasing absolute ambiguity aversion; they are a special case of the act-dependent variational model of Proposition 1.

We interpret the representation (7) in light of the risk mitigation story.¹⁷ Two forces contribute to the evaluation of the act f . On the one hand, the decision maker optimal action induces the most favorable model among the ones justifying f over the outside option. This model, $q \in C_f$, determines the benchmark probability for the relative entropy $R(\cdot \| q)$. On the other hand, due to concerns for misspecification, the decision maker employs the worst model $p \in \Delta(S)$ to compute the expected utility, trading-off an higher value of the relative entropy $R(p \| q)$ the more p diverges from q . ◀

5 Relative ambiguity attitudes

We now study the impact of a *relative* change in the proportion of the certainty part of an act. In particular, we characterize preferences displaying decreasing relative ambiguity

¹⁷The conflicting selves narrative would apply as well with obvious adjustments. We employ it later to interpret the model of Example 4 concerning decreasing relative ambiguity aversion.

aversion. Recall that $x_* \in X$ denotes the worst consequence, that is, $y \succsim x_*$ for all $y \in X$.

AXIOM (decreasing relative ambiguity aversion). *For all $f \in \mathcal{F}$, $y \in X$, and $\alpha, \beta \in (0, 1)$, if $\alpha \leq \beta$, then*

$$\alpha f + (1 - \alpha)x_* \succsim \alpha y + (1 - \alpha)x_* \implies \beta f + (1 - \beta)x_* \succsim \beta y + (1 - \beta)x_*.$$

This axiom says that if an act $\alpha f + (1 - \alpha)x_*$ is preferred to a constant act $\alpha y + (1 - \alpha)x_*$, where both acts can be expressed as mixtures with the worst consequence, then such a ranking is preserved after decreasing the proportion associated with the worst consequence in both acts. It modifies the axiom proposed by Xue (2020) by restricting to mixtures involving the worst consequence only.

We are ready to state our main representation for decreasing relative ambiguity aversion. It mirrors the representation of decreasing absolute ambiguity aversion in Theorem 1, the main distinctions being: (i) constant superadditivity of the aggregators is replaced by positive superhomogeneity, and (ii) we require the existence of a worst consequence.

THEOREM 2. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) *\succsim is MBA, admits a worst consequence, and exhibits decreasing relative ambiguity aversion.*
- (ii) *There exist an affine function $u : X \rightarrow \mathbb{R}$ with $\min u(X) = 0$ and a linearly continuous family \mathcal{G} of monotone, positively superhomogeneous in the first argument, and quasiconvex functions $G : \mathbb{R} \times \Delta(S) \rightarrow \bar{\mathbb{R}}$ such that*

$$f \succsim g \iff \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int u(f) dp, p\right) \geq \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int u(g) dp, p\right)$$

for all $f, g \in \mathcal{F}$ and $\max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G(\int u(x) dp, p) = u(x)$ for all $x \in X$.

The two interpretations provided for Lemma 1 and Theorem 1—the dual-self and the risk mitigation interpretation—apply to Theorem 2 as well. In particular, through the positive superhomogeneity of each $G(\cdot, p)$, for all $p \in \Delta(S)$, we can view decreasing relative ambiguity aversion as a property of the aggregators only.

Analogously to Example 1, the following connects the representation of Theorem 2 to second-order expected utility with risk mitigation when the function governing ambiguity attitudes satisfies decreasing relative risk aversion (DRRA), i.e., the Arrow-Pratt coefficient of relative risk aversion is decreasing.

EXAMPLE 3 (second-order expected utility with risk mitigation; DRRA). Consider again the preferences introduced in Example 1. In particular, for $f \in \mathcal{F}$,

$$V(f) = \max_{q \in Q} I_q(u(f)) = \max_{q \in Q} \phi^{-1} \left(\int \phi(u(f)) dq \right)$$

where $Q \subseteq \Delta^\sigma(S)$ is a finite set of models, $u(X) = [0, \infty)$, and ϕ is a continuous, strictly increasing, and concave function. By Lemma 11 and 12 in Appendix B.3, if ϕ is also twice differentiable and DRRA, i.e., $t \mapsto -t\phi''(t)/\phi'(t)$ is decreasing, then each I_q is positively superhomogeneous.¹⁸ As a result, these preferences satisfy decreasing relative ambiguity aversion and can be represented following Theorem 2, where each G_q , defined as in Example 1, is positively superhomogeneous. ◀

REMARK 2. We show that, under ambiguity aversion, our formulation of decreasing relative ambiguity aversion is equivalent to the smooth ambiguity functional (Klibanoff et al., 2005) satisfying DRRA. Formally, for $f \in \mathcal{F}$, the smooth certainty equivalent is

$$V(f) = \phi^{-1} \left(\int \phi \left(\int u(f) dp \right) d\mu \right)$$

where we assume that $u(X) = [1, \infty)$, μ is a countably additive probability measure over $\Delta^\sigma(S)$, (S, Σ) is a measurable space and Σ is non-trivial, and $\phi : [1, \infty) \rightarrow \mathbb{R}$ is continuous, strictly increasing, and concave, capturing ambiguity aversion. Proposition 19 in Appendix B.4 shows that, if ϕ is twice differentiable, then smooth ambiguity preferences satisfy decreasing relative ambiguity aversion for all countably additive probability measures μ over $\Delta^\sigma(S)$ if and only if ϕ is DRRA. ◀

Under the same assumptions of Theorem 2, the following result provides a different representation for decreasing relative ambiguity aversion. It shows that this property admits an act-dependent confidence preference representation. In their standard formulation (Chateauneuf and Faro, 2009), these preferences are characterized by a confidence function $d : \Delta(S) \rightarrow [0, \infty]$ associating each probability model to its relative confidence level. In our case, the set of confidence functions varies for each act.

PROPOSITION 2. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) *\succsim is MBA, admits a worst consequence, and exhibits decreasing relative ambiguity aversion.*

¹⁸For instance, the function $\phi : t \mapsto t + \sqrt{t}$ satisfies all the listed hypotheses.

(ii) There exist an affine function $u : X \rightarrow \mathbb{R}$, with $\min u(X) = 0$, and, for all $f \in \mathcal{F}$, a set D_f of upper semicontinuous and quasiconcave $d : \Delta(S) \rightarrow [0, \infty]$ such that

$$f \succsim g \iff \max_{d \in D_f} \min_{p \in \Delta(S)} \frac{\int_S u(f) dp}{d(p)} \geq \max_{d \in D_g} \min_{p \in \Delta(S)} \frac{\int_S u(g) dp}{d(p)}$$

where $\max_{d \in D_x} \min_{p \in \Delta(S)} u(x)/d(p) = u(x)$ for all $x \in X$, and $D_f \subseteq D_g$ for all $f, g \in \mathcal{F}$ with $u(f) \leq u(g)$.

This result generalizes the existing representations for homothetic preferences mentioned above and mirrors the act-dependent variational model of Proposition 1. Notice, positive superhomogeneity of the aggregators (Theorem 2) is captured by the set of confidence functions enlarging as the utility levels increase. Therefore, the decision maker displays an higher degree of relative confidence for higher utility levels.

The following example relates to the representation of decreasing relative ambiguity aversion of Proposition 2 by considering act-dependent entropic-confidence preferences with outside option.

EXAMPLE 4 (confidence preferences with outside option). Following Example 2, we model a decision maker that can mitigate the uncertainty of each act $f \in \mathcal{F}$ by exerting effort whenever it is valuable to do so. Formally, to each act $f \in \mathcal{F}$ is associated the set

$$D_f = \left\{ q \in Q : \int u(f) dq \geq \theta \right\} \cup \{q_u\}$$

where $Q \subseteq \Delta(S)$ is a finite set of models, S is finite, $\Sigma = 2^S$, $\theta \in [0, \infty)$, and q_u denotes the uniform distribution over S . Notice that, for $f, g \in \mathcal{F}$, $u(f) \leq u(g)$ implies $D_f \subseteq D_g$.

We define act-dependent entropic-confidence preferences as follows

$$V(f) = \max_{q \in D_f} \min_{p \in \Delta(S)} \frac{\int u(f) dp}{\exp(-R(p\|q))} \quad (8)$$

where, as before, $R(\cdot\|\cdot)$ denotes the relative entropy. These preferences satisfy decreasing relative ambiguity aversion; they are a special case of the act-dependent confidence preferences of Proposition 2.

We interpret the representation (8) by following the narrative of the intrapersonal game between two conflicting selves. To evaluate the act f , Optimism selects the most favorable model among the ones justifying f over the outside option. This choice, $q \in D_f$,

determines the benchmark probability to compute the relative entropy $R(\cdot \| q)$. As a result, Pessimism faces a trade-off: the choice of the pessimistic model $p \in \Delta(S)$ determines the expected utility of the agent, but, at the same time, the more it diverges from q , the higher the value of the relative entropy $R(p \| q)$, and of the ratio in (8). ◀

6 A risk sharing application

In this section, we apply our representations of changing ambiguity attitudes to an exchange economy with a single consumption good and no aggregate uncertainty. We want to investigate whether it is efficient for agents displaying general ambiguity preferences to take bets. We show that, under regularity requirements, the joint combination of decreasing absolute ambiguity aversion and increasing relative ambiguity aversion, when at least one of them is non-constant, implies *strict pseudoconcavity at certainty*. Under this condition, as shown by Ghirardato and Siniscalchi (2018a), betting is inefficient if and only if the agents share at least one *supporting probability*,¹⁹ i.e., the agents' beliefs that support the consumption bundle have a non-empty intersection.

Non-constant changing ambiguity attitudes. We introduce non-constant absolute and relative ambiguity attitudes, which follow from the notions already discussed with obvious adjustments.

AXIOM (non-constant decreasing absolute ambiguity aversion). *For all $f \in \mathcal{F}$, $x, y, z \in X$, and $\alpha \in (0, 1)$, if $y \succ x$, then*

$$\alpha f + (1 - \alpha)x \succsim \alpha z + (1 - \alpha)x \implies \alpha f + (1 - \alpha)y \succ \alpha z + (1 - \alpha)y.$$

Recall, if it exists, $x_* \in X$ denotes the worst consequence, that is, $y \succ x_*$ for all $y \in X$.

AXIOM (non-constant increasing relative ambiguity aversion). *For all $f \in \mathcal{F}$, $y \in X$, and $\alpha, \beta \in (0, 1)$, if $\alpha > \beta$, then*

$$\alpha f + (1 - \alpha)x_* \succsim \alpha y + (1 - \alpha)x_* \implies \beta f + (1 - \beta)x_* \succ \beta y + (1 - \beta)x_*.$$

Lemma 13 in Appendix B.5 shows that, under MBA preferences, decreasing absolute

¹⁹We adopt the terminology of Ghirardato and Siniscalchi (2018a), although Rigotti et al. (2008) first introduced this notion under the name of *subjective beliefs*.

ambiguity aversion and increasing relative ambiguity aversion hold simultaneously if and only if the certainty equivalent functional satisfies both constant superadditivity and positive subhomogeneity.²⁰ Furthermore, whenever either one of the two behavioral properties holds non-constantly, the corresponding functional property holds strict.

The economy. We model an exchange economy populated by finitely many agents $N := \{1, \dots, N\}$. Let the state space S be finite as well. Each agent, indexed by $i \in N$, has a utility function $V_i : \mathbb{R}_+^S \rightarrow \mathbb{R}$ and is dispensed with an endowment $\omega_i \in \mathbb{R}_+^S$. For simplicity, we abstract away from risk attitudes and restrict the attention to the case of risk neutrality, i.e., $V_i(x) = x$ for every $x \in \mathbb{R}_+$. Finally, as in Ghirardato and Siniscalchi (2018a), this economy features no aggregate uncertainty, i.e., $\sum_i \omega_i = \bar{\omega}$ for some $\bar{\omega} > 0$.

An *allocation* is a vector $(f_1, \dots, f_N) \in \mathbb{R}_+^{N \times S}$ where each f_i is the consumption bundle assigned to agent $i \in N$ contingent on each state. We say that an allocation is *feasible* if $\sum_i f_i = \bar{\omega}$; *interior* if, for all $i \in N$, $f_i > 0$; *full-insurance* if, for all $i \in N$, $f_i = x_i$ for some $x_i \in \mathbb{R}_+$; *Pareto-efficient* if it is feasible, and there is no other feasible allocation (g_1, \dots, g_N) such that $V_i(g_i) \geq V_i(f_i)$ for all $i \in N$, and $V_j(g_j) > V_j(f_j)$ for some $j \in N$; a *competitive equilibrium with transfers* if it is feasible, and there exist prices $q \in \mathbb{R}_{++}^S$ and transfers $(T_i)_{i \in N} \in \mathbb{R}_+^N$ with $\sum_i T_i = 0$ such that $f_i \in \arg \max_{\{g \in \mathbb{R}_+^S : q \cdot g \leq q \cdot \omega_i + T_i\}} V_i(g)$ for all $i \in N$.

Supporting probabilities and strict pseudoconcavity. For every $i \in N$, the set of *supporting probabilities* at an allocation $f \in \mathbb{R}_+^S$ is

$$\pi_i(f) = \left\{ p \in \Delta(S) : \forall g \in \mathbb{R}_+^S, V_i(g) \geq V_i(f) \implies p \cdot g \geq p \cdot f \right\}.$$

Notice that, $\pi_i(f)$ can be interpreted as the set of (normalized) prices such that any bundle weakly preferred to f is at least as costly as f .

Following Ghirardato and Siniscalchi (2018a), a function $V : \mathbb{R}_+^S \rightarrow \mathbb{R}$ is *strictly pseudoconcave at* $f \in \mathbb{R}_{++}^S$ if, for all $g \neq f$,

$$V(g) \geq V(f) \implies \forall q \in \partial V(f), \quad q \cdot (g - f) > 0,$$

²⁰In the context of recursive ambiguity models, Lemma 1 in Strzalecki (2013) shows that constant superadditive and positive subhomogeneous functionals represent preferences for early resolution of uncertainty.

where $\partial V(f)$ denotes the Clarke subdifferential of V at f (Clarke, 1983).²¹ Furthermore, V is *strict pseudoconcavity at certainty* if it is strictly pseudoconcave at x for all $x \in \mathbb{R}_{++}$.

Changing ambiguity attitudes and risk sharing. As in Ghirardato and Siniscalchi (2018a), we say that $V : \mathbb{R}_+^S \rightarrow \mathbb{R}$ is *nice* if it is locally Lipschitz, strictly monotone and, for every $x \in \mathbb{R}_{++}$, continuously differentiable in a neighborhood of x with $\nabla V(x) \neq 0$, where $\nabla V(x)$ denotes the gradient of V at x . We are ready to state the main result of this section.

PROPOSITION 3. *If $V : \mathbb{R}_+^S \rightarrow \mathbb{R}$ is normalized, nice, and satisfies constant superadditivity and positive subhomogeneity, with at least one being strict, then V satisfies strict pseudoconcavity at certainty.*

By applying Proposition 3 and Theorem 3 in Ghirardato and Siniscalchi (2018a), we retrieve the following equivalences, which generalize the risk sharing result of Rigotti et al. (2008). Thus, we establish a connection between risk sharing and our general representations of changing ambiguity attitudes.

COROLLARY 1. *For each $i \in N$, assume V_i is nice, and satisfies constant superadditivity and positive subhomogeneity, with at least one being strict. The following are equivalent*

- (i) *Every Pareto-efficient allocation is a full-insurance allocation.*
- (ii) *Every feasible, full-insurance allocation is Pareto-efficient.*
- (iii) *For every feasible, full-insurance allocation (x_1, \dots, x_N) , $\bigcap_{i \in N} \pi_i(x_i) \neq \emptyset$.*

Furthermore, under the above equivalent conditions, every interior, feasible, full-insurance allocation is a competitive equilibrium with transfers.

In contrast to the previous result, decreasing relative ambiguity aversion appears at odds with risk sharing. Intuitively, by Theorem 2, this property implies that preferences are “convex at zero,” that is, for every $f \in \mathcal{F}$ and $\alpha \in (0, 1)$,

$$V(\alpha f + (1 - \alpha)0) \leq \alpha V(f) + (1 - \alpha)V(0).$$

²¹For $n \geq 1$, an open subset $B \subseteq \mathbb{R}^n$, and a function $V : B \rightarrow \mathbb{R}$, the *Clarke subdifferential* of V at $b \in B$ is

$$\partial V(b) = \text{cl conv} \left\{ \lim_{k \rightarrow \infty} d^k : \exists (b^k) \rightarrow b \text{ such that } d^k = \nabla V(b^k), \forall k \right\},$$

where cl conv denotes the closure of the convex hull, and $\nabla V(b^k)$ the gradient of V at b^k .

The following example considers an economy where each V_i is nice but satisfies positive superhomogeneity as well as constant additivity. We construct a full-insurance allocation that is not Pareto-efficient. This suggests that agents displaying decreasing relative ambiguity aversion may be willing to bet even if they share the same beliefs.

EXAMPLE 5. Let $S = \{s_1, s_2\}$, $N = \{1, 2\}$, and $\omega_1 = \omega_2 = (1/2, 1/2)$. Define $V_1 = V_2 := V$ as

$$V(f) = \max_{H \in \Psi} H(f),$$

where $\Psi = \{H_1, H_2, H_3\}$, and

$$\begin{aligned} H_1(f) &= 1/9 \cdot f(s_1) + 8/9 \cdot f(s_2) - 0.1 \\ H_2(f) &= \frac{1}{10} \log \left(\frac{1}{2} \left(e^{10 \left(\frac{1}{4} f(s_1) + \frac{3}{4} f(s_2) \right)} \right) + \frac{1}{2} \left(e^{10 \left(\frac{3}{4} f(s_1) + \frac{1}{4} f(s_2) \right)} \right) \right) \\ H_3(f) &= 8/9 \cdot f(s_1) + 1/9 \cdot f(s_2) - 0.1. \end{aligned}$$

Since $V_1 = V_2$ the agents share the same beliefs at the initial full-insurance allocation. However, we show that such allocation is not Pareto-efficient which implies that Corollary 1 does not hold under non-constant decreasing relative ambiguity aversion. To this end, notice that, V is monotone, normalized, and continuous. Therefore, by Proposition 1 in Cerreia-Vioglio et al. (2011a), V represents MBA preferences. Furthermore, since each $H \in \Psi$ is positively superhomogeneous, V is positively superhomogeneous as well, and, by Theorem 2, decreasing relative ambiguity aversion holds. Finally, V is also nice.

The initial full-insurance endowment is not Pareto-efficient. Indeed, the feasible allocation $((0.4, 0.6), (0.6, 0.4))$ achieves a strictly higher level of utility for both agents:

$$H_1((1/2, 1/2)) = 0.4, \quad H_2((1/2, 1/2)) = 0.5, \quad H_3((1/2, 1/2)) = 0.4,$$

while

$$\begin{aligned} H_1((0.4, 0.6)) &= H_3((0.6, 0.4)) = 0.4\bar{7} \\ H_2((0.4, 0.6)) &= H_2((0.6, 0.4)) = 0.512 \\ H_3((0.4, 0.6)) &= H_1((0.6, 0.4)) = 0.3\bar{2}, \end{aligned}$$

which implies that $V_1((0.4, 0.6)) = V_2((0.6, 0.4)) = 0.512 > V((1/2, 1/2)) = 0.5$. ◀

Appendix

The Appendix is organized as follows. In Appendix A, we present the proofs of some mathematical results we later employ in Appendix B to prove the results in the main text.

A Mathematical appendix

A.1 Toolkit lemmas

In this part of the appendix we provide characterizations of constant superadditivity which we later use to prove our results in the main text.

LEMMA 2. *Fix $K \subseteq \mathbb{R}$. A map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$ is constant superadditive if and only if $I(\varphi + k) \leq I(\varphi) + k$ for all $\varphi \in B_0(\Sigma, K)$ and $k \leq 0$ such that $\varphi + k \in B_0(\Sigma, K)$.*

Proof. Suppose I is constant superadditive, $\varphi \in B_0(\Sigma, K)$, and $k \leq 0$ such that $\varphi + k \in B_0(\Sigma, K)$. It follows that

$$I(\varphi) = I(\varphi + k - k) \geq I(\varphi + k) - k.$$

The converse is proved in an analogous fashion. □

LEMMA 3. *Fix a convex $K \subseteq \mathbb{R}$ with $0 \in \text{int}K$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. We have that I is constant superadditive if and only if*

$$I(\alpha\varphi + (1 - \alpha)k) \geq I(\alpha\varphi) + (1 - \alpha)k \tag{9}$$

for all $\alpha \in [0, 1]$, $\varphi \in B_0(\Sigma, K)$, and $k \in K \cap \mathbb{R}_+$.

Proof. The proof follows from a minor modification of Lemma 5 in Cerreia-Vioglio et al. (2014), we report it here for completeness. If I is constant superadditive, then it is straightforward to see that (9) must hold. As for the converse, first notice that, since I is continuous, we can assume that $K = (a, b)$ for some $a, b \in \bar{\mathbb{R}}$ without loss of generality. Let $\varphi \in B_0(\Sigma, K)$ and $k \in \mathbb{R}_{++}$ be such that $\varphi + k \in B_0(\Sigma, K)$. Then, we have that $a < \inf\varphi, \sup(\varphi + k) < b$ and there exists $\alpha \in (0, 1)$ such that $\varphi/\alpha, (\varphi + k)/\alpha \in B_0(\Sigma, K)$. Since $k > 0$ and $0 \in \text{int}K = (a, b)$, there exists $n \geq 2$ such that

$$\frac{\frac{1}{n}k}{1 - \alpha} \in K.$$

Then, by convexity of K ,

$$\frac{\varphi + \frac{m}{n}k}{\alpha} = \frac{m}{n} \frac{\varphi + k}{\alpha} + \left(1 - \frac{m}{n}\right) \frac{\varphi}{\alpha} \in B_0(\Sigma, K)$$

for all $m = 0, \dots, n-1$. For all $m = 0, \dots, n-1$, by (9), it follows that

$$I\left(\varphi + \frac{m+1}{n}k\right) = I\left(\alpha \frac{\varphi + \frac{m}{n}k}{\alpha} + (1-\alpha) \frac{\frac{1}{n}k}{1-\alpha}\right) \geq I\left(\varphi + \frac{m}{n}k\right) + \frac{k}{n}.$$

Moreover, this yields,

$$I(\varphi + k) - I(\varphi) = \sum_{m=0}^{n-1} \left(I\left(\varphi + \frac{m+1}{n}k\right) - I\left(\varphi + \frac{m}{n}k\right) \right) \geq \sum_{m=0}^{n-1} \frac{k}{n} = k.$$

Thus, I is constant superadditive. □

For all $T : B_0(\Sigma, K) \rightarrow \mathbb{R}$, $\varphi \in B_0(\Sigma, K)$, define $\bar{T} : B_0(\Sigma, -K) \rightarrow \mathbb{R}$ as $\bar{T}(\varphi) = -T(-\varphi)$.

LEMMA 4. *Fix $K \subseteq \mathbb{R}$ and a map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, I is constant superadditive if and only if \bar{I} is constant superadditive.*

Proof. Suppose I is constant superadditive. If $\varphi \in B_0(\Sigma, -K)$, $k \leq 0$, and $\varphi + k \in B_0(\Sigma, -K)$, then

$$\bar{I}(\varphi + k) = -I(-\varphi - k) \leq -I(-\varphi) + k = \bar{I}(\varphi) + k.$$

Since $k \leq 0$, by Lemma 2, \bar{I} is constant superadditive. The converse follows by $I = \bar{\bar{I}}$. □

A.2 Extension results

Here we provide some instrumental extension results. For each interval $K \subseteq \mathbb{R}$, we denote by $K_\infty := K \cup [\sup K, \infty)$.

LEMMA 5. *If K is an interval and $T : B_0(\Sigma, K) \rightarrow \mathbb{R}$ is constant superadditive, monotone, and normalized, then the map $\tilde{T} : B_0(\Sigma, K_\infty) \rightarrow \bar{\mathbb{R}}$ defined as*

$$\tilde{T}(\psi) = \sup \{ T(\varphi) + m : \varphi \in B_0(\Sigma, K), m \geq 0, \varphi + m \leq \psi \}$$

for all $\psi \in B_0(\Sigma, K_\infty)$ is a real-valued, constant superadditive, monotone, and normalized extension of T to $B_0(\Sigma, K_\infty)$.

Proof. Monotonicity is immediate as the larger ψ gets the larger is the set over which we are taking the supremum. To prove that \tilde{T} extends T notice that for all $\psi \in B_0(\Sigma, K)$, we have $\tilde{T}(\psi) \geq T(\psi)$. Conversely, notice that for all $\varphi, \psi \in B_0(\Sigma, K)$, and $m \geq 0$, with $\varphi + m \leq \psi$, we have that $\varphi + m \in B_0(\Sigma, K)$ and by monotonicity and constant superadditivity of T , we have

$$T(\varphi) + m \leq T(\varphi + m) \leq T(\psi).$$

By the arbitrariness of φ and m , we have $\tilde{T}(\psi) = T(\psi)$. Thus, \tilde{T} is a monotonic extension of T . Now we prove that \tilde{T} satisfies constant superadditivity. To this end notice first that

$$\tilde{T}(\psi) = \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi} \left\{ T(\varphi) + \inf_{s \in S} \{ \psi(s) - \varphi(s) \} \right\}$$

for all $\psi \in B_0(\Sigma, K_\infty)$. Let $\psi \in B_0(\Sigma, K_\infty)$ and $k \geq 0$,

$$\begin{aligned} \tilde{T}(\psi + k) &= \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi + k} \left\{ T(\varphi) + \inf_{s \in S} \{ \psi(s) + k - \varphi(s) \} \right\} \\ &\geq \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi} \left\{ T(\varphi) + \inf_{s \in S} \{ \psi(s) + k - \varphi(s) \} \right\} \\ &= \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi} \left\{ T(\varphi) + \inf_{s \in S} \{ \psi(s) - \varphi(s) \} \right\} + k \\ &= \tilde{T}(\psi) + k. \end{aligned}$$

Next we prove that \tilde{T} is a real-valued map. If $\psi \in B_0(\Sigma, K)$, then $\tilde{T}(\psi) = T(\psi) \in \mathbb{R}$. If $\psi \in B_0(\Sigma, K_\infty) \setminus B_0(\Sigma, K)$, then for all $\varphi \in B_0(\Sigma, K)$ and $m \geq 0$ with $\varphi + m \leq \psi$, we have that

$$T(\varphi) + m \leq T(\sup \varphi) + m = \sup \varphi + m \leq \sup \psi < \infty$$

the equality follows from the normalization of T and the fact that φ is a simple function.

To conclude we show that \tilde{T} is also normalized. If $k \in K$, then $\tilde{T}(k) = T(k) = k$. If $k \in K_\infty \setminus K$, then $k \geq \sup K$. Then, there exists $t \in K$ and $m \geq 0$ such that $t + m = k$, and hence $\tilde{T}(k) = T(t) + m = t + m = k$. Thus, \tilde{T} is normalized. \square

LEMMA 6. *If K is an interval with $\inf K = 0$ and $T : B_0(\Sigma, K) \rightarrow \mathbb{R}$ is positively superhomogeneous, monotone, and normalized, then the map $\tilde{T} : B_0(\Sigma, K_\infty) \rightarrow \bar{\mathbb{R}}$ defined as*

$$\tilde{T}(\psi) = \sup \{ \alpha T(\varphi) : \varphi \in B_0(\Sigma, K), \alpha \geq 1, \alpha \varphi \leq \psi \}$$

for all $\psi \in B_0(\Sigma, K_\infty)$ is a positively superhomogeneous, monotone, and normalized extension of T to $B_0(\Sigma, K_\infty)$.

Proof. Monotonicity is immediate as the larger ψ gets the larger is the set over which we are taking the supremum. To prove that \tilde{T} extends T notice that for all $\psi \in B_0(\Sigma, K)$, we have $\tilde{T}(\psi) \geq T(\psi)$. Conversely, notice that for all $\psi, \varphi \in B_0(\Sigma, K)$, and $\alpha \geq 1$, with $\alpha\varphi \leq \psi$, we have that $\alpha\varphi \in B_0(\Sigma, K)$ and by monotonicity and positive superhomogeneity of T , we have

$$\alpha T(\varphi) \leq T(\alpha\varphi) \leq T(\psi).$$

By the arbitrariness of φ and α , we have $\tilde{T}(\psi) = T(\psi)$. Thus, \tilde{T} is a monotonic extension of T . Now we prove that \tilde{T} satisfies positive superhomogeneity. To this end notice first that

$$\tilde{T}(\psi) = \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi} \left\{ T(\varphi) \inf_{s \in S: \varphi(s) > 0} \left\{ \frac{\psi(s)}{\varphi(s)} \right\} \right\}$$

for all $\psi \in B_0(\Sigma, K_\infty)$. Let $\psi \in B_0(\Sigma, K_\infty)$ and $\alpha \geq 1$,

$$\begin{aligned} \tilde{T}(\alpha\psi) &= \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \alpha\psi} \left\{ T(\varphi) \inf_{s \in S: \varphi(s) > 0} \left\{ \frac{\alpha\psi(s)}{\varphi(s)} \right\} \right\} \\ &\geq \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi} \left\{ T(\varphi) \inf_{s \in S: \varphi(s) > 0} \left\{ \frac{\alpha\psi(s)}{\varphi(s)} \right\} \right\} \\ &= \alpha \sup_{\varphi \in B_0(\Sigma, K), \varphi \leq \psi} \left\{ T(\varphi) \inf_{s \in S: \varphi(s) > 0} \left\{ \frac{\psi(s)}{\varphi(s)} \right\} \right\} \\ &= \alpha \tilde{T}(\psi). \end{aligned}$$

Next we prove that \tilde{T} is a real-valued map. If $\psi \in B_0(\Sigma, K)$, then $\tilde{T}(\psi) = T(\psi) \in \mathbb{R}$. If $\psi \in B_0(\Sigma, K_\infty) \setminus B_0(\Sigma, K)$, then for all $\varphi \in B_0(\Sigma, K)$ and $\alpha \geq 1$ with $\alpha\varphi \leq \psi$, we have that

$$\alpha T(\varphi) \leq \alpha T(\sup \varphi) = \alpha \sup \varphi \leq \sup \psi < \infty$$

the equality follows from the normalization of T and the fact that φ is a simple function.

To conclude we prove that \tilde{T} is normalized. If $k \in K$, then $\tilde{T}(k) = T(k) = k$. If $k \in K_\infty \setminus K$, then $k \geq \sup K$. Therefore, there exist $\alpha \geq 1$ and $t \in K$ such that $\alpha t = k$, and hence $\tilde{T}(k) = \alpha T(t) = \alpha t = k$. \square

A.3 Envelope representations

Before going into further results, we provide notation used throughout the remainder of Appendix A. For a convex set $K \subseteq \mathbb{R}$, and $\xi, \varphi \in B_0(\Sigma, K)$, we define the following sets

$$C_\xi(\varphi) = \{k \in \mathbb{R} : \xi + k \geq \varphi, \xi + k \in B_0(\Sigma, K)\}, \quad M_\xi(\varphi) = \{\alpha > 0 : \alpha\xi \geq \varphi, \alpha\xi \in B_0(\Sigma, K)\}$$

$$c_\xi(\varphi) = \{k \in \mathbb{R} : \varphi \geq \xi + k, \xi + k \in B_0(\Sigma, K)\}, \quad m_\xi(\varphi) = \{\alpha > 0 : \varphi \geq \alpha\xi, \alpha\xi \in B_0(\Sigma, K)\}.$$

All the sets defined above could well be empty. Fixing a map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$, we define the following auxiliary functionals associated to I :

$$I_\xi(\varphi) = \inf\{I(\xi + k) : k \in C_\xi(\varphi)\}, \quad J_\xi(\varphi) = \inf\{I(\alpha\xi) : \alpha \in M_\xi(\varphi)\},$$

$$S_\xi(\varphi) = \sup\{I(\xi + k) : k \in c_\xi(\varphi)\}, \quad H_\xi(\varphi) = \sup\{I(\alpha\xi) : \alpha \in m_\xi(\varphi)\}$$

for all $\xi, \varphi \in B_0(\Sigma, K)$. We adopt the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. If $C_\xi(\varphi) = M_\xi(\varphi) = \emptyset$, then $I_\xi(\varphi) = J_\xi(\varphi) = \infty$, while if $c_\xi(\varphi) = m_\xi(\varphi) = \emptyset$, $S_\xi(\varphi) = H_\xi(\varphi) = -\infty$.

PROPOSITION 4. *Fix a convex $K \subseteq \mathbb{R}$ and a monotone and normalized map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. Then, for all $\xi \in B_0(\Sigma, K)$,*

1. I_ξ, J_ξ, S_ξ , and H_ξ are monotone.
2. $I_\xi \geq I, J_\xi \geq I, S_\xi \leq I$, and $H_\xi \leq I$.
3. I_ξ is quasiconvex and S_ξ is quasiconcave.
4. If $K \subseteq \mathbb{R}_+$, then J_ξ is quasiconvex and H_ξ is quasiconcave.

Proof. 1. For all $\varphi, \psi, \xi \in B_0(\Sigma, K)$ with $\varphi \geq \psi$ we have $C_\xi(\varphi) \subseteq C_\xi(\psi)$, $M_\xi(\varphi) \subseteq M_\xi(\psi)$, $c_\xi(\varphi) \subseteq c_\xi(\psi)$, and $m_\xi(\varphi) \subseteq m_\xi(\psi)$. This implies that I_ξ, J_ξ, S_ξ and H_ξ are monotone.

2. Fix $\varphi, \xi \in B_0(\Sigma, K)$. If $C_\xi(\varphi) = \emptyset$, $I_\xi(\varphi) = \infty \geq I(\varphi)$, while $J_\xi(\varphi) = \infty \geq I(\varphi)$ whenever $M_\xi(\varphi) = \emptyset$, and similarly if $c_\xi(\varphi) = \emptyset$, then $I(\varphi) \geq -\infty = S_\xi(\varphi)$, while $I(\varphi) \geq -\infty = H_\xi(\varphi)$ whenever $m_\xi(\varphi) = \emptyset$. Thus, we assume such sets to be nonempty. By monotonicity of I , we have that $I(\xi + k) \geq I(\varphi)$ for all $k \in C_\xi(\varphi)$. Analogously, $I(\alpha\xi) \geq I(\varphi)$ for all $\alpha \in M_\xi(\varphi)$. Thus, $I_\xi(\varphi) \geq I(\varphi)$ and $J_\xi(\varphi) \geq I(\varphi)$ for all $\varphi \in B_0(\Sigma, K)$. An analogous argument for $k \in c_\xi(\varphi)$ and $\alpha \in m_\xi(\varphi)$ concludes the proof of the claim.

3. Fix $\xi \in B_0(\Sigma, K)$, $\alpha \in (0, 1)$, and $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$. If $\max\{I_\xi(\varphi_1), I_\xi(\varphi_2)\} = \infty$, then $I_\xi(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leq \max\{I_\xi(\varphi_1), I_\xi(\varphi_2)\}$. Thus, assume that $I_\xi(\varphi_1), I_\xi(\varphi_2)$ are finite. We prove that for all $t \in \mathbb{R}$,²²

$$(\forall i = 1, 2, I_\xi(\varphi_i) \leq t) \implies I_\xi(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leq t.$$

For all $\varepsilon > 0$, there exist $k_1, k_2 \in \mathbb{R}$ such that, for all $i = 1, 2$, $\xi + k_i \geq \varphi_i$, $\xi + k_i \in B_0(\Sigma, K)$, and

$$I(\xi + k_i) \leq I_\xi(\varphi_i) + \varepsilon \leq t + \varepsilon.$$

We have that $\alpha\varphi_1 + (1-\alpha)\varphi_2 \leq \alpha(\xi + k_1) + (1-\alpha)(\xi + k_2) \leq \xi + k$ where $k = \max\{k_1, k_2\}$ and hence, by monotonicity of I ,

$$I_\xi(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leq I(\xi + k) \leq t + \varepsilon.$$

The claim follows by arbitrariness of ε . The quasiconcavity of S_ξ is proved analogously.

4. Fix $\xi \in B_0(\Sigma, K)$, $\alpha \in (0, 1)$, and $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$. If $\max\{J_\xi(\varphi_1), J_\xi(\varphi_2)\} = \infty$, then $J_\xi(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leq \max\{J_\xi(\varphi_1), J_\xi(\varphi_2)\}$. Thus, assume that $J_\xi(\varphi_1), J_\xi(\varphi_2)$ are finite. Now we pass to the quasiconvexity of J_ξ . We prove that for all $t \in \mathbb{R}$,

$$(\forall i = 1, 2, J_\xi(\varphi_i) \leq t) \implies J_\xi(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leq t.$$

For all $\varepsilon > 0$, there are $\beta_1, \beta_2 > 0$ so that, for all $i = 1, 2$, $\beta_i\xi \geq \varphi_i$, $\beta_i\xi \in B_0(\Sigma, K)$, and

$$I(\beta_i\xi) \leq J_\xi(\varphi_i) + \varepsilon \leq t + \varepsilon.$$

Since $K \subseteq \mathbb{R}_+$, we have that $\xi \geq 0$ and hence $\alpha\varphi_1 + (1-\alpha)\varphi_2 \leq \alpha\beta_1\xi + (1-\alpha)\beta_2\xi \leq \beta\xi$ where $\beta = \max\{\beta_1, \beta_2\}$. Therefore, by monotonicity of I ,

$$J_\xi(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leq I(\beta\xi) \leq t + \varepsilon.$$

By the arbitrariness of ε the claim follows. The quasiconcavity of H_ξ is proved analogously.

²²Here we follow the same steps provided in Han et al. (2022) (proof of Theorem 4.1 therein).

□

PROPOSITION 5. *Fix a convex $K \subseteq \mathbb{R}$ and a monotone map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. If I is constant superadditive, then, for all $\xi \in B_0(\Sigma, K)$, I_ξ and S_ξ are constant superadditive.*

Proof. Let $\varphi, \xi \in B_0(\Sigma, K)$, and $m \geq 0$ be such that $\varphi + m \in B_0(\Sigma, K)$. If $I_\xi(\varphi + m) = \infty$, then the claim follows. Thus, suppose $I_\xi(\varphi + m) \in \mathbb{R}$. By Proposition 4 and the monotonicity of I , we have that I_ξ is monotone, and hence $I_\xi(\varphi) \in \mathbb{R}$. These simple observations imply that $C_\xi(\varphi + m)$ and $C_\xi(\varphi)$ are not empty. Then, we have that

$$\begin{aligned} I_\xi(\varphi + m) &= \inf\{I(\xi + k) : k \in \mathbb{R} \text{ s.t. } \xi + k \in B_0(\Sigma, K) \text{ and } \xi + (k - m) \geq \varphi\} \\ &= \inf\{I(\xi + (k - m) + m) : k \in \mathbb{R} \text{ s.t. } \xi + k \in B_0(\Sigma, K) \text{ and } \xi + (k - m) \geq \varphi\} \\ &\geq \inf\{I(\xi + k - m) : k \in \mathbb{R} \text{ s.t. } \xi + k \in B_0(\Sigma, K) \text{ and } \xi + (k - m) \geq \varphi\} + m \\ &\geq \inf\{I(\xi + r) : r \in \mathbb{R} \text{ s.t. } \xi + r \in B_0(\Sigma, K) \text{ and } \xi + r \geq \varphi\} + m \\ &= I_\xi(\varphi) + m \end{aligned}$$

where the second-to-last inequality follows from the constant superadditivity of I and the fact that since K is convex and $m \geq 0$, the following inequalities

$$\varphi \leq \xi + k - m \leq \xi + k$$

imply that $\xi + k - m \in B_0(\Sigma, K)$, whenever $\xi + k \in B_0(\Sigma, K)$. The last inequality follows from observing that for all $k \in \mathbb{R}$ such that $\xi + k \in B_0(\Sigma, K)$ and $\xi + k - m \geq \varphi$, there is $r \in \mathbb{R}$ such that $\xi + r \in B_0(\Sigma, K)$ and $\xi + r \geq \varphi$. The proof of the constant superadditivity of S_ξ is analogous and follows the same steps choosing $m \leq 0$ and applying Lemma 2. □

Denote by \mathcal{K} the collection of intervals K in \mathbb{R} which satisfy at least one of the following: $K \in \{[0, a], (0, a], [0, a), (0, a), [b, 0], (b, 0], [b, 0), (b, 0) : a > 0, b < 0\}$; $0 \in \text{int}K$; $K = [0, \infty)$, $K = (-\infty, 0]$.

PROPOSITION 6. *Fix $K \in \mathcal{K}$ and a monotone map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. If I is positively superhomogeneous, then, for all $\xi \in B_0(\Sigma, K)$, J_ξ is positively superhomogeneous. If $K = [0, \infty)$, then H_ξ is positively superhomogeneous.*

Proof. Let $\lambda \geq 1$ and $\varphi, \xi \in B_0(\Sigma, K)$ with $\lambda\varphi \in B_0(\Sigma, K)$. If $J_\xi(\lambda\varphi) = \infty$, then the claim follows. If $J_\xi(\lambda\varphi) \in \mathbb{R}$, then there exists $\alpha > 0$ such that $\alpha\xi \geq \lambda\varphi$ and $\alpha\xi \in B_0(\Sigma, K)$.

Since $K \in \mathcal{K}$ and $\lambda \geq 1$, we have that $(\alpha/\lambda)\xi \in B_0(\Sigma, K)$. Thus, $J_\xi(\varphi) \in \mathbb{R}$. These simple observations imply that $M_\xi(\lambda\varphi)$ and $M_\xi(\varphi)$ are not empty. Then, we have that

$$\begin{aligned} J_\xi(\lambda\varphi) &= \inf\{I(\alpha\xi) : \alpha > 0 \text{ s.t. } \alpha\xi \geq \lambda\varphi \text{ and } \alpha\xi \in B_0(\Sigma, K)\} \\ &= \inf\left\{I\left(\alpha\lambda\frac{\xi}{\lambda}\right) : \alpha > 0 \text{ s.t. } \frac{\alpha}{\lambda}\xi \geq \varphi \text{ and } \alpha\xi \in B_0(\Sigma, K)\right\} \\ &\geq \lambda \inf\left\{I\left(\alpha\frac{\xi}{\lambda}\right) : \alpha > 0 \text{ s.t. } \frac{\alpha}{\lambda}\xi \geq \varphi \text{ and } \alpha\xi \in B_0(\Sigma, K)\right\} \\ &\geq \lambda \inf\{I(\gamma\xi) : \gamma > 0 \text{ s.t. } \gamma\xi \geq \varphi \text{ and } \gamma\xi \in B_0(\Sigma, K)\} = \lambda J_\xi(\varphi) \end{aligned}$$

where the second-to-last inequality follows from positive superhomogeneity and the fact that $(\alpha/\lambda)\xi \in B_0(\Sigma, K)$ whenever $\alpha\xi \in B_0(\Sigma, K)$. The last inequality follows from the fact that, for each $\alpha > 0$, with $(\alpha/\lambda)\xi \geq \varphi$ and $\alpha\xi \in B_0(\Sigma, K)$, there exists $\gamma > 0$ such that $\gamma\xi \geq \varphi$ and $\gamma\xi \in B_0(\Sigma, K)$. Thus, J_ξ is positively superhomogeneous.

If $K = [0, \infty)$, the exact same arguments but with $\lambda \in (0, 1)$ such that $\lambda\varphi \in B_0(\Sigma, K)$, yield that $H_\xi(\lambda\varphi) \leq \lambda H_\xi(\varphi)$, and hence H_ξ is positively superhomogeneous. The condition $K = [0, \infty)$ is used to guarantee that $(\alpha/\lambda)\xi \in B_0(\Sigma, K)$. \square

A.4 Semicontinuity

For each $\varphi, \psi \in B_0(\Sigma)$, the set $[\varphi, \psi] = \{\xi \in B_0(\Sigma) : \varphi \leq \xi \leq \psi\}$ will be referred to as an *order interval*. A subset Y of $B_0(\Sigma, K)$ is *lower open* (resp., *upper open*) if, for all $\varphi \in Y$, there exists $\varepsilon > 0$ such that $[\varphi - \varepsilon, \varphi] \subseteq Y$ (resp., $[\varphi, \varphi + \varepsilon] \subseteq Y$). Fix $K \subseteq \mathbb{R}$. It is important to notice that whenever K is a nontrivial interval, $B_0(\Sigma, K)$ is either lower open, or upper open, or an order interval. Given a function $T : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ we define $T^- : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ as

$$T^-(\varphi) = \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} T(\psi)$$

for all $\varphi \in B_0(\Sigma, K)$, where $\mathcal{N}(\varphi, K)$ denotes the set of all neighborhoods of φ .²³ For simplicity, when $K = \mathbb{R}$ we write $\mathcal{N}(\varphi)$. Such function T^- is referred to as the *lower*

²³The neighborhoods of $\varphi \in B_0(\Sigma, K)$ are the subsets of $B_0(\Sigma, K)$ that contain φ in their interior, relative to $(B_0(\Sigma, K), \|\cdot\|_\infty)$.

semicontinuous envelope of T . By Proposition 3.5 in Dal Maso (1993), we have that

$$\{T^- \leq k\} = \bigcap_{t > k} \text{cl}(\{T \leq t\}) \quad (10)$$

for all $k \in \mathbb{R}$. Analogously, we define the *upper semicontinuous envelope* of T as the function $T^+ : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ such that $T^+(\varphi) = \inf_{U \in \mathcal{N}(\varphi, K)} \sup_{\psi \in U} T(\psi)$ for all $\varphi \in B_0(\Sigma, K)$. We report some basic properties of such envelopes, the first two points extend Lemmas 29 and 30 in Cerreia-Vioglio et al. (2011c).

LEMMA 7. *Let $K \subseteq \mathbb{R}$ be convex and fix a map $T : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. Then,*

- (i) *If T is monotone and $B_0(\Sigma, K)$ is lower open, then T^- is monotone.*
- (ii) *If T is monotone and $B_0(\Sigma, K)$ is either upper open or an order interval, then T^+ is monotone.*
- (iii) *If T is quasiconvex (quasiconcave), then T^- and T^+ are quasiconvex (quasiconcave).*
- (iv) *If T is constant superadditive, then T^- and T^+ are constant superadditive.*
- (v) *If T is positively superhomogeneous, then T^- and T^+ are positively superhomogeneous.*

Proof. We show that (i) and (ii) hold. Lemma 29 and 30 of Cerreia-Vioglio et al. (2011c) can be easily adapted to show that, when $B_0(\Sigma, K)$ is lower open, then $T^-(\varphi) = \sup_n T(\varphi_n)$ for all $\varphi, \varphi_n \in B_0(\Sigma, K)$ and $\varphi_n \rightarrow \varphi$ with $\varphi > \varphi_n$. Analogously, when $B_0(\Sigma, K)$ is upper open, then $T^+(\varphi) = \inf_n T(\varphi_n)$ for all $\varphi, \varphi_n \in B_0(\Sigma, K)$ and $\varphi_n \rightarrow \varphi$ with $\varphi < \varphi_n$. Then, in these cases, monotonicity of T^- and T^+ readily follows from the same proofs as in Lemmas 29 and 30 of Cerreia-Vioglio et al. (2011c).²⁴ The only remaining case is the one with $K = [a, b]$ for some $a < b$. To this end, define $\hat{T} : B_0(\Sigma) \rightarrow \bar{\mathbb{R}}$ as

$$\hat{T}(\varphi) = T(\varphi \wedge b \vee a)$$

for all $\varphi \in B_0(\Sigma)$. Clearly, \hat{T} is a monotonic extension of T . We want to show that \hat{T}^+ is an extension of T^+ . If this is the case, by the monotonicity of \hat{T} , Lemma 29 of Cerreia-Vioglio et al. (2011c) would yield that T^+ must be monotone. Fix $\varphi \in B_0(\Sigma, K)$. For all $U \in \mathcal{N}(\varphi)$,

²⁴Their proofs are provided in the working paper version of their paper.

we have that

$$\sup_{\psi \in U} \hat{T}(\psi) \geq \sup_{\psi \in U \cap B_0(\Sigma, K)} \hat{T}(\psi) = \sup_{\psi \in U \cap B_0(\Sigma, K)} T(\psi)$$

which yields

$$\begin{aligned} \hat{T}^+(\varphi) &= \inf_{U \in \mathcal{N}(\varphi)} \sup_{\psi \in U} \hat{T}(\psi) \geq \inf_{U \in \mathcal{N}(\varphi)} \sup_{\psi \in U \cap B_0(\Sigma, K)} T(\psi) \\ &= \inf_{O \in \mathcal{N}(\varphi, K)} \sup_{\psi \in O} T(\psi) = T^+(\varphi). \end{aligned}$$

To prove the converse inequality, notice that, by Proposition 3.6(b) in Dal Maso (1993), there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $B_0(\Sigma)$ such that $\varphi_n \rightarrow \varphi$ and

$$\hat{T}^+(\varphi) \leq \liminf_{n \rightarrow \infty} \hat{T}(\varphi_n). \quad (11)$$

Then, by definition of \hat{T} , we have that

$$\begin{aligned} \hat{T}^+(\varphi) &\leq \liminf_{n \rightarrow \infty} \hat{T}(\varphi_n) = \liminf_{n \rightarrow \infty} T(\varphi_n \wedge b \vee a) \\ &\leq \limsup_{n \rightarrow \infty} T(\varphi_n \wedge b \vee a) \leq \limsup_{n \rightarrow \infty} T^+(\varphi_n \wedge b \vee a) \leq T^+(\varphi) \end{aligned}$$

where the first inequality is due to (11), the second to definition of \hat{T} , the third to the properties of limit superiors, the fourth to the fact that $T^+ \geq T$, and the last one to the fact that $\varphi_n \wedge b \vee a \rightarrow \varphi$ and that T^+ is upper semicontinuous. Thus, we have that $\hat{T}^+(\varphi) = T^+(\varphi)$.

Point (iii) follows from (10) and the analogous version for T^+ . To prove point (iv), let $\varphi \in B_0(\Sigma, K)$ and $k \geq 0$ with $\varphi + k \in B_0(\Sigma, K)$. Notice that $\mathcal{N}(\varphi + k, K) = \{U + k : U \in \mathcal{N}(\varphi, K)\}$. Thus,

$$\begin{aligned} T^-(\varphi + k) &= \sup_{U + k \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U + k} T(\psi) = \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} T(\psi + k) \\ &\geq \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} T(\psi) + k = T^-(\varphi) + k. \end{aligned}$$

The proof for T^+ is analogous. To prove point (v) let $\varphi \in B_0(\Sigma, K)$ and $\alpha \geq 1$ such that

$\alpha\varphi \in B_0(\Sigma, K)$. Notice that $\mathcal{N}(\alpha\varphi, K) = \{\alpha U : U \in \mathcal{N}(\varphi, K)\}$. Thus,

$$\begin{aligned} T^-(\alpha\varphi) &= \sup_{\alpha U \in \mathcal{N}(\alpha\varphi, K)} \inf_{\psi \in \alpha U} T(\psi) = \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} T(\alpha\psi) \\ &\geq \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} \alpha T(\psi) = \alpha T^-(\varphi). \end{aligned}$$

The proof for T^+ is analogous. □

PROPOSITION 7. *Let $K \subseteq \mathbb{R}$ be convex and $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ a monotone map. The following hold*

(i) *For all $\varphi \in B_0(\Sigma, K)$,*

$$I(\varphi) = \min_{\xi \in B_0(\Sigma, K)} I_{\xi}^-(\varphi) = \min_{\xi \in B_0(\Sigma, K)} J_{\xi}^-(\varphi) = \max_{\xi \in B_0(\Sigma, K)} S_{\xi}^-(\varphi) = \max_{\xi \in B_0(\Sigma, K)} H_{\xi}^-(\varphi)$$

provided that $B_0(\Sigma, K)$ is lower open and I lower semicontinuous.

(ii) *For all $\varphi \in B_0(\Sigma, K)$,*

$$I(\varphi) = \min_{\xi \in B_0(\Sigma, K)} I_{\xi}^+(\varphi) = \min_{\xi \in B_0(\Sigma, K)} J_{\xi}^+(\varphi) = \max_{\xi \in B_0(\Sigma, K)} S_{\xi}^+(\varphi) = \max_{\xi \in B_0(\Sigma, K)} H_{\xi}^+(\varphi)$$

provided that $B_0(\Sigma, K)$ is either upper open or an order interval and I upper semicontinuous.

Proof. Let $\varphi \in B_0(\Sigma, K)$. By Proposition 4, we have that

$$I_{\xi}^-(\varphi) \geq \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} I(\psi) = I(\varphi)$$

for all $\varphi, \xi \in B_0(\Sigma, K)$, where the last equality follows by the continuity of I . Moreover,

$$I_{\varphi}^-(\varphi) = \sup_{U \in \mathcal{N}(\varphi, K)} \inf_{\psi \in U} I_{\varphi}(\psi) \leq \sup_{U \in \mathcal{N}(\varphi, K)} I_{\varphi}(\varphi) = I(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$, where the last equality follows from the fact that $I_{\varphi}(\varphi) = I(\varphi)$. The exact same proof can be provided to retrieve the rest of the equalities. □

A.5 Quasiconvex duality

Given $K \subseteq \mathbb{R}$ and $T : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$, we define the function $G_T : \mathbb{R} \times \Delta(S) \rightarrow \bar{\mathbb{R}}$ as

$$G_T(t, p) = \sup \left\{ T(\varphi) : \varphi \in B_0(\Sigma, K) \text{ and } \int_S \varphi dp \leq t \right\}$$

for all $(t, p) \in \mathbb{R} \times \Delta(S)$. It is immediate to see that G_T is monotone in the first argument. Moreover, by Lemma 31 in Cerreia-Vioglio et al. (2011c), G_T is quasiconvex.

PROPOSITION 8. *If $K \subseteq \mathbb{R}$ is convex and unbounded from above and $T : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ is constant superadditive, then G_T is constant superadditive in the first argument.*

Proof. The proof mimics proof methods provided in Cerreia-Vioglio et al. (2022). Fix $t \in \mathbb{R}$, $k \geq 0$, and $p \in \Delta(S)$. By definition of $G_T(t, p)$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $B_0(\Sigma, K)$ such that $T(\varphi_n) \uparrow G_T(t, p)$ and $\int_S \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. Then, we have that $\int_S \varphi_n + k \leq t + k$. Since K is unbounded from above, we have that $\varphi_n + k \in B_0(\Sigma, K)$ for all $n \in \mathbb{N}$. Given that T is constant superadditive, we have

$$G_T(t + k, p) \geq T(\varphi_n + k) \geq T(\varphi_n) + k \rightarrow G_T(t, p) + k$$

proving that G_T is constant superadditive in the first argument. \square

PROPOSITION 9. *If either $K = (a, \infty)$, $K = [b, \infty)$, or $K = \mathbb{R}$ for some $a > 0$ $b \geq 0$, and $T : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ is positively superhomogeneous, then G_T is positively superhomogeneous in the first argument.*

Proof. Fix $t \in \mathbb{R}$, $\alpha \geq 1$, and $p \in \Delta(S)$. By definition of $G_T(t, p)$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $B_0(\Sigma, K)$ such that $T(\varphi_n) \uparrow G_T(t, p)$ and $\int_S \varphi_n dp \leq t$ for all $n \in \mathbb{N}$. Then, we have that $\int_S \alpha \varphi_n \leq \alpha t$. By the hypotheses on K , we have that $\alpha \varphi_n \in B_0(\Sigma, K)$ for all $n \in \mathbb{N}$. Given that T is positively superhomogeneous, we have

$$G_T(\alpha t, p) \geq T(\alpha \varphi_n) \geq \alpha T(\varphi_n) \rightarrow \alpha G_T(t, p)$$

proving that G_T is positively superhomogeneous in the first argument. \square

The exact same approach also yields the following.

LEMMA 8. *If $K = [0, \infty)$ and $T : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$ is positively homogeneous, then G_T is positively homogeneous.*

Proof. Fix $t \in \mathbb{R}$, $\alpha > 0$, and $p \in \Delta(S)$. By definition of $G_T(t, p)$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $B_0(\Sigma, K)$ such that $T(\varphi_n) \uparrow G_T(t, p)$ and $\int_S \varphi_n d p \leq t$ for all $n \in \mathbb{N}$. Then, we have that $\int_S \alpha \varphi_n \leq \alpha t$. Since $K = [0, \infty)$, we have that $\alpha \varphi_n \in B_0(\Sigma, K)$ for all $n \in \mathbb{N}$. Moreover, since T is positively homogeneous we have that

$$G_T(\alpha t, p) \geq T(\alpha \varphi_n) = \alpha T(\varphi_n) \rightarrow \alpha G_T(t, p)$$

proving that G_T is positively homogeneous in the first argument. \square

A.6 Monotone and normalized functionals

Section A.5 provided the tools to retrieve the following general representation.

PROPOSITION 10. *Fix a convex $K \subseteq \mathbb{R}$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. Then, the following are equivalent*

- (i) *I is monotone and normalized.*
- (ii) *There exists a family Φ of monotone and quasiconvex functionals mapping from $B_0(\Sigma, K)$ to $\bar{\mathbb{R}}$ such that*

$$I(\varphi) = \min_{J \in \Phi} J(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\min_{J \in \Phi} J(k) = k$ for all $k \in K$. If $B_0(\Sigma, K)$ is lower open, then each J can be taken lower semicontinuous. If $B_0(\Sigma, K)$ is upper open or an order interval, then each J can be taken upper semicontinuous.

- (iii) *There exists a family Ψ of monotone and quasiconcave functionals mapping from $B_0(\Sigma, K)$ to $\bar{\mathbb{R}}$ such that*

$$I(\varphi) = \max_{H \in \Psi} H(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\max_{H \in \Psi} H(k) = k$ for all $k \in K$. If $B_0(\Sigma, K)$ is lower open, then each H can be taken lower semicontinuous. If $B_0(\Sigma, K)$ is upper open or an order interval, then each H can be taken upper semicontinuous.

Proof. (i) implies (ii). Suppose first that $B_0(\Sigma, K)$ is lower open. Let $\Phi = \{I_{\xi}^- : \xi \in B_0(\Sigma, K)\}$. By Proposition 4 and Lemma 7, we have that each $J \in \Phi$ is monotone, quasiconvex, and lower semicontinuous. Moreover, by Proposition 7, we have that, for all $\varphi \in B_0(\Sigma, K)$,

$$I(\varphi) = \min_{J \in \Phi} J(\varphi).$$

Since I is normalized, $\min_{J \in \Phi} J(k) = k$ for all $k \in K$. If $B_0(\Sigma, K)$ is upper open or an order interval the same steps applied to the family $\Phi = \{I_\xi^+ : \xi \in B_0(\Sigma, K)\}$, yield the claim with upper semicontinuity.

(i) implies (iii). We apply the same steps as in the previous implication, but with $\Psi = \{S_\xi^- : \xi \in B_0(\Sigma, K)\}$, when $B_0(\Sigma, K)$ is lower open, or with $\Psi = \{S_\xi^+ : \xi \in B_0(\Sigma, K)\}$, when $B_0(\Sigma, K)$ is either upper open or an order interval. By Proposition 4 and Lemma 7, we have that each $H \in \Psi$ is monotone, quasiconcave, and lower semicontinuous or upper semicontinuous depending on whether $B_0(\Sigma, K)$ is lower open, upper open, or an order interval. Moreover, by Proposition 7, we have that, for all $\varphi \in B_0(\Sigma, K)$,

$$I(\varphi) = \max_{H \in \Psi} H(\varphi).$$

Since I is normalized, $\max_{H \in \Psi} H(k) = k$ for all $k \in K$.

To conclude, notice that the implications: (ii) implies (i) and (iii) implies (i) follow immediately from the monotonicity of all $J \in \Phi$ and $H \in \Psi$, and the normalization from $\min_{J \in \Phi} J(k) = k$ and $\max_{H \in \Psi} H(k) = k$ for all $k \in K$. \square

A.7 Constant superadditivity

PROPOSITION 11. *Fix a convex $K \subseteq \mathbb{R}$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and constant superadditive.*
- (ii) *There exists a family Φ of monotone, constant superadditive, and quasiconvex functionals mapping from $B_0(\Sigma, K)$ to $\bar{\mathbb{R}}$ such that*

$$I(\varphi) = \min_{J \in \Phi} J(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\min_{J \in \Phi} J(k) = k$ for all $k \in K$. If $B_0(\Sigma, K)$ is lower open, then each J can be taken lower semicontinuous. If $B_0(\Sigma, K)$ is upper open or an order interval, then each J can be taken upper semicontinuous.

- (iii) *There exists a family Ψ of monotone, constant superadditive, and quasiconcave functionals mapping from $B_0(\Sigma, K)$ to $\bar{\mathbb{R}}$ such that*

$$I(\varphi) = \max_{H \in \Psi} H(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\max_{H \in \Psi} H(k) = k$ for all $k \in K$. If $B_0(\Sigma, K)$ is lower open, then each H can be taken lower semicontinuous. If $B_0(\Sigma, K)$ is upper open or an order interval, then each H can be taken upper semicontinuous.

Proof. It is immediate to see that both (ii) and (iii) imply (i). We prove that (i) implies (ii). By Proposition 7, if K is lower open, then

$$I(\varphi) = \min_{\xi \in B_0(\Sigma, K)} I_{\xi}^{-}(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. If K is upper open or an order interval, the same result holds with I_{ξ}^{+} in place of I_{ξ}^{-} for all $\xi \in B_0(\Sigma, K)$. Moreover, by Propositions 4, 5, and Lemma 7 each I_{ξ}^{-}, I_{ξ}^{+} , depending on whether K is lower open, upper open, or an order interval, is monotone, quasiconvex, and constant superadditive. Thus, letting either $\Phi = \{I_{\xi}^{-} : \xi \in B_0(\Sigma, K)\}$ or $\Phi = \{I_{\xi}^{+} : \xi \in B_0(\Sigma, K)\}$ according to the cases mentioned above, the claim follows. To conclude the proof it is sufficient to show that (i) implies (iii). The proof is totally analogous to the previous steps with either

$$I(\varphi) = \max_{\xi \in B_0(\Sigma, K)} S_{\xi}^{-}(\varphi) \text{ or } I(\varphi) = \max_{\xi \in B_0(\Sigma, K)} S_{\xi}^{+}(\varphi)$$

in the case where K is lower open, upper open, or an order interval. In particular, we let $\Psi = \{S_{\xi}^{-} : \xi \in B_0(\Sigma, K)\}$ when K is lower open. Instead if K is upper open or an order interval we set $\Psi = \{S_{\xi}^{+} : \xi \in B_0(\Sigma, K)\}$. By Propositions 4, 5, and Lemma 7 each S_{ξ}^{-}, S_{ξ}^{+} , depending on whether K is lower open, upper open, or an order interval, is monotone, quasiconcave, and constant superadditive. \square

PROPOSITION 12. *Fix a convex $K \subseteq \mathbb{R}$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and constant superadditive.*
- (ii) *There exists a family \mathcal{G} of constant superadditive and increasing in the first argument, and quasiconvex functions $G : \mathbb{R} \times \Delta(S) \rightarrow \mathbb{R}$ such that*

$$I(\varphi) = \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int_S \varphi dp, p\right)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G(k, p) = k$ for all $k \in K$.

Proof. It is immediate to see that (ii) implies (i). To prove the converse, notice by Lemma 5, I admits a monotone, normalized, and constant superadditive extension \tilde{I} . Moreover, we have two cases:

- If K is lower open, then \tilde{I}^- , by the continuity of I and Lemma 7, is a lower semi-continuous, monotone, and constant superadditive extension of I on $B_0(\Sigma, K_\infty)$. Then, the result follows from Proposition 11, Theorem 36 in Cerreia-Vioglio et al. (2011c),²⁵ Proposition 8, and the normalization of $I = \tilde{I}^+|_{B_0(\Sigma, K)}$.
- If K is upper open or an order interval, then \tilde{I}^+ , by the continuity of I and Lemma 7, is an upper semicontinuous, monotone, and constant superadditive extension of I on $B_0(\Sigma, K_\infty)$. Then, the result follows from Proposition 11, Theorem 36 in Cerreia-Vioglio et al. (2011c), Proposition 8, and the normalization of $I = \tilde{I}^+|_{B_0(\Sigma, K)}$.

Thus, the proof is concluded. \square

The following result is adapted from Proposition A.3 in Han et al. (2022).

PROPOSITION 13. *Fix a convex $K \subseteq \mathbb{R}$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and constant superadditive.*
- (ii) *For all $\varphi \in B_0(\Sigma, K)$, there exists a family Φ_φ of monotone, constant additive, and convex functionals, mapping from $B_0(\Sigma, K)$ to \mathbb{R} , such that*

$$I(\varphi) = \min_{J \in \Phi_\varphi} J(\varphi).$$

Moreover, $\min_{J \in \Phi_\varphi} J(k) = k$ for all $k \in K$, and $\Phi_{\varphi_1} \subseteq \Phi_{\varphi_2}$ for all $\varphi_1 \geq \varphi_2$ in $B_0(\Sigma, K)$.

- (iii) *For all $\varphi \in B_0(\Sigma, K)$, there exists a family Ψ_φ of monotone, constant additive, and concave functionals, mapping from $B_0(\Sigma, K)$ to \mathbb{R} , such that*

$$I(\varphi) = \max_{H \in \Psi_\varphi} H(\varphi).$$

Moreover, $\max_{H \in \Psi_\varphi} H(k) = k$ for all $k \in K$, and $\Psi_{\varphi_1} \subseteq \Psi_{\varphi_2}$ for all $\varphi_2 \geq \varphi_1$ in $B_0(\Sigma, K)$.

²⁵Theorem 36 in Cerreia-Vioglio et al. (2011c) is provided for a function mapping in \mathbb{R} , but the exact same proof they provide applies to extended real-valued maps.

Proof. We start proving that (i) implies (ii). Let $\varphi \in B_0(\Sigma, K)$. For all $\xi \in B_0(\Sigma, K)$ with $\xi \geq \varphi$, we have that for all $s \in S$,

$$\xi(s) \geq \xi(s) + \sup\{\varphi - \xi\} \geq \xi(s) + \varphi(s) - \xi(s) = \varphi(s).$$

Thus, since K is convex, for all $\xi \in B_0(\Sigma, K)$ with $\xi \geq \varphi$ it follows that $\xi + \sup\{\varphi - \xi\} \in B_0(\Sigma, K)$ and $\sup\{\varphi - \xi\} \leq 0$. By monotonicity and constant superadditivity of I ,

$$I(\varphi) \leq \min_{\xi \geq \varphi} I(\xi + \sup\{\varphi - \xi\}) \leq \min_{\xi \geq \varphi} I(\xi) + \sup\{\varphi - \xi\} \leq I(\varphi).$$

For all $\xi, \psi \in B_0(\Sigma, K)$, define $T_\xi(\psi) = I(\xi) + \sup\{\psi - \xi\}$. Therefore, letting $\Phi_\varphi = \{T_\xi : \xi \in B_0(\Sigma, K), \xi \geq \varphi\}$, we have that for all $\varphi \in B_0(\Sigma, K)$,

$$I(\varphi) = \min_{J \in \Phi_\varphi} J(\varphi).$$

Notice that each $J \in \Phi_\varphi$ is monotone, constant additive, and quasiconvex. Thus, each $J \in \Phi_\varphi$ is also convex. Moreover, as I is normalized, $\min_{J \in \Phi_k} J(k) = k$ for all $k \in \mathbb{R}$. Clearly, $\Phi_{\varphi_1} \subseteq \Phi_{\varphi_2}$ for all $\varphi_1 \geq \varphi_2$ in $B_0(\Sigma, K)$.

(i) implies (iii). By Lemma 4, \bar{I} is constant superadditive. Since it is also monotone and normalized, we have that

$$-I(-\varphi) = \min_{J \in \Phi_\varphi} J(\varphi)$$

for all $\varphi \in B_0(\Sigma)$ and some family of monotone, constant additive, and convex functionals. Thus, we have that $I(\varphi) = \max_{J \in \Phi_{-\varphi}} \bar{J}(\varphi)$ for all $\varphi \in B_0(\Sigma)$. Letting $\Psi_\varphi = \{\bar{J} : J \in \Phi_{-\varphi}\}$ the claim follows. In particular, notice that

$$\varphi_1 \leq \varphi_2 \implies \Psi_{\varphi_1} \subseteq \Psi_{\varphi_2}$$

for all $\varphi_1, \varphi_2 \in B_0(\Sigma)$.

(iii) implies (i). Monotonicity and normalization are straightforward. We only need to prove that I is constant superadditive. To this end, let $\varphi \in B_0(\Sigma, K)$ and $k \geq 0$ with $\varphi + k \in B_0(\Sigma, K)$. By assumption, we have $\Psi_\varphi \subseteq \Psi_{\varphi+k}$, and hence

$$I(\varphi + k) = \max_{J \in \Psi_{\varphi+k}} J(\varphi + k) = \max_{J \in \Psi_{\varphi+k}} J(\varphi) + k \geq \max_{J \in \Psi_\varphi} J(\varphi) + k = I(\varphi) + k.$$

The proof that (ii) implies (i) is proved analogously upon choosing $k \leq 0$. \square

Therefore, applying convex duality results we retrieve the following representation. Han et al. (2022) provides a specific form for the penalty functions, the interested reader can consult their Appendix A.

PROPOSITION 14. *Fix a convex $K \subseteq \mathbb{R}$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and constant superadditive.*
- (ii) *For all $\varphi \in B_0(\Sigma, K)$, there exists a family C_φ of lower semicontinuous and convex functionals $c : \Delta(S) \rightarrow (-\infty, \infty]$ such that*

$$I(\varphi) = \max_{c \in C_\varphi} \min_{p \in \Delta(S)} \left\{ \int_S \varphi dp + c(p) \right\}.$$

Moreover, $\max_{c \in C_k} \min_{p \in \Delta(S)} c(p) = 0$ and $C_{\varphi_1} \subseteq C_{\varphi_2}$ for all $\varphi_2 \geq \varphi_1$ in $B_0(\Sigma, K)$.

Proof. The result follows from Proposition 13 above and Proposition 4 and Theorem 3 in Cerreia-Vioglio et al. (2014). \square

A.8 Positive superhomogeneity

Here, we report the representation results for positive superhomogeneity.

PROPOSITION 15. *Fix a convex $K \subseteq \mathbb{R}$ with $\min K = 0$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \bar{\mathbb{R}}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and positively superhomogeneous.*
- (ii) *There exists a family Φ of monotone, positively superhomogeneous, upper semicontinuous, and quasiconvex functionals mapping from $B_0(\Sigma, K)$ to $\bar{\mathbb{R}}$ such that*

$$I(\varphi) = \min_{J \in \Phi} J(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\min_{J \in \Phi} J(k) = k$ for all $k \in K$.

- (iii) *There exists a family Ψ of monotone, positively superhomogeneous, upper semicontinuous, and quasiconcave functionals mapping from $B_0(\Sigma, K)$ to $\bar{\mathbb{R}}$ such that*

$$I(\varphi) = \max_{H \in \Psi} H(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\max_{H \in \Psi} H(k) = k$ for all $k \in K$.

Proof. It is immediate to see that both (ii) and (iii) imply (i). If we prove that (i) implies (ii) and that (i) implies (iii), then the proof is concluded. We start with (i) implies (ii). First notice that by Lemma 6, there exists a positively superhomogeneous, monotone, and normalized extension \tilde{I} of I on $B_0(\Sigma, K_\infty)$. Since I is continuous, and K is either upper open or an order interval, we have that \tilde{I}^+ is an upper semicontinuous, positively superhomogeneous, and monotone extension of I on $B_0(\Sigma, K_\infty)$. By Proposition 7, since K_∞ is upper open and \tilde{I}^+ is an extension of I , we have that

$$I(\varphi) = \tilde{I}^+(\varphi) = \min_{\xi \in B_0(\Sigma, K_\infty)} J_\xi^+(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$, where the J_ξ 's are the auxiliary functionals associated to \tilde{I}^+ . By Propositions 4, 6, and Lemma 7, J_ξ^+ is monotone, positively superhomogeneous, upper semicontinuous, and quasiconvex. Thus, letting $\Phi = \{J_\xi^+ : \xi \in B_0(\Sigma, K_\infty)\}$ according to the cases mentioned above, the claim follows. By the normalization of I , we have that $\min_{J \in \Phi} J(k) = \tilde{I}^+(k) = I(k) = k$ for all $k \in K$.

To prove that (i) implies (iii), we can apply the same exact reasoning to the family $\Psi = \{H_\xi^+ : \xi \in B_0(\Sigma, K_\infty)\}$ of auxiliary functionals associated to the extension \tilde{I}^+ of I . In particular, by Proposition 7, since K_∞ is upper open and \tilde{I}^+ is an extension of I , we have that

$$I(\varphi) = \tilde{I}^+(\varphi) = \max_{\xi \in B_0(\Sigma, K_\infty)} H_\xi^+(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. By Propositions 4, 6, and Lemma 7 each H_ξ^+ is monotone, upper semicontinuous, quasiconvex, and positively superhomogeneous. Thus, letting $\Psi = \{H_\xi^+ : \xi \in B_0(\Sigma, K_\infty)\}$, the claim follows. By the normalization of I , we have that $\max_{H \in \Psi} H(k) = \tilde{I}^+(k) = I(k) = k$ for all $k \in K$. \square

PROPOSITION 16. *Fix a convex $K \subseteq \mathbb{R}$ with $\min K = 0$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and positively superhomogeneous.*
- (ii) *There exists a family \mathcal{G} of positively superhomogeneous and increasing in the first argument, and quasiconvex functions $G : \mathbb{R} \times \Delta(S) \rightarrow \mathbb{R}$ such that*

$$I(\varphi) = \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int_S \varphi dp, p\right)$$

for all $\varphi \in B_0(\Sigma, K)$. Moreover, $\max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G(k, p) = k$ for all $k \in K$.

Proof. That (ii) implies (i) is immediate to see. To prove the converse, notice by Lemma 6, I admits a monotone, normalized, and positively superhomogeneous extension \tilde{I} . Moreover, as K is either upper open or an order interval we have that \tilde{I}^+ , by the continuity of I and Lemma 7, is an upper semicontinuous, monotone, and positively superhomogeneous extension of I on $B_0(\Sigma, K_\infty)$. Then, the result follows from Proposition 15, Theorem 36 in Cerreia-Vioglio et al. (2011c),²⁶ Proposition 9, and the normalization of $I = \tilde{I}^+|_{B_0(\Sigma, K)}$. \square

PROPOSITION 17. *Fix a convex $K \subseteq \mathbb{R}$ with $\min K = 0$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and positively superhomogeneous.*
- (ii) *For all $\varphi \in B_0(\Sigma, K)$, there exists a family Φ_φ of monotone, sublinear functionals, mapping from $B_0(\Sigma, K)$ to \mathbb{R} , such that*

$$I(\varphi) = \min_{J \in \Phi_\varphi} J(\varphi).$$

Moreover, $\min_{J \in \Phi_\varphi} J(k) = k$ for all $k \in K$, and $\Phi_{\varphi_1} \subseteq \Phi_{\varphi_2}$ for all $\varphi_1 \geq \varphi_2$ in $B_0(\Sigma, K)$.

- (iii) *For all $\varphi \in B_0(\Sigma, K)$, there exists a family Ψ_φ of monotone, superlinear functionals, mapping from $B_0(\Sigma, K)$ to \mathbb{R} , such that*

$$I(\varphi) = \max_{H \in \Psi_\varphi} H(\varphi).$$

Moreover, $\max_{H \in \Psi_\varphi} H(k) = k$ for all $k \in K$, and $\Psi_{\varphi_1} \subseteq \Psi_{\varphi_2}$ for all $\varphi_2 \geq \varphi_1$ in $B_0(\Sigma, K)$.

Proof. We start showing that (i) implies (ii). By Lemma 6 we have that I admits a positively superhomogeneous, and monotone extension \tilde{I} on $B_0(\Sigma, K_\infty)$. Moreover, since I is continuous, and K is either upper open or an order interval, Lemma 7 yields that \tilde{I}^+ is an upper semicontinuous, positively superhomogeneous, and monotone extension of I . Now we provide the representation result directly for \tilde{I}^+ . Notice that here $K_\infty = [0, \infty)$. Fix $\varphi \in B_0(\Sigma, K_\infty)$. For all $\xi \geq \varphi$ in $B_0(\Sigma, K_\infty)$ with $\xi > 0$, we have $\sup\{\varphi/\xi\} \leq 1$, thus by

²⁶Theorem 36 in Cerreia-Vioglio et al. (2011c) is provided for a function mapping in \mathbb{R} , but the exact same proof they provide applies to extended real-valued maps.

monotonicity and positively superhomogeneity of I ,

$$\tilde{I}^+(\varphi) \leq \min_{\xi > 0, \xi \geq \varphi} \tilde{I}^+ \left(\xi \sup \left\{ \frac{\varphi}{\xi} \right\} \right) \leq \min_{\xi > 0, \xi \geq \varphi} \tilde{I}^+(\xi) \sup \left\{ \frac{\varphi}{\xi} \right\} \leq \tilde{I}^+ \left(\varphi + \frac{1}{n} \right) \sup \left\{ \frac{\varphi}{\varphi + \frac{1}{n}} \right\} \leq \tilde{I}^+ \left(\varphi + \frac{1}{n} \right)$$

for all $n \in \mathbb{N}$. By upper semicontinuity of \tilde{I}^+ it follows that, for all $\varphi \in B_0(\Sigma, K_\infty)$,

$$I(\varphi) = \limsup_{n \rightarrow \infty} \tilde{I}^+(\varphi) \leq \limsup_{n \rightarrow \infty} \min_{\xi > 0, \xi \geq \varphi} \tilde{I}^+(\xi) \sup \left\{ \frac{\varphi}{\xi} \right\} \leq \limsup_{n \rightarrow \infty} \tilde{I}^+ \left(\varphi + \frac{1}{n} \right) \leq I(\varphi)$$

Therefore, for all $\varphi \in B_0(\Sigma, K)$,

$$I(\varphi) = \tilde{I}^+(\varphi) = \min_{\xi > 0, \xi \geq \varphi} I(\xi) \sup \left\{ \frac{\varphi}{\xi} \right\}.$$

For all $\xi \in B_0(\Sigma, K_\infty)$ with $\xi > 0$, define $T_\xi(\psi) = I(\xi) \sup\{\psi/\xi\}$ for all $\psi \in B_0(\Sigma, K_\infty)$. Therefore, letting $\Phi_\varphi = \{T_\xi : \xi \in B_0(\Sigma, K_\infty), \xi > 0, \xi \geq \varphi\}$, we have that

$$I(\varphi) = \min_{J \in \Phi_\varphi} J(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Since $I(\xi) \geq I(0) = 0$ for all $\xi \in B_0(\Sigma, K)$, we have that each J is monotone and sublinear. Clearly, $\Phi_{\varphi_1} \subseteq \Phi_{\varphi_2}$ for all $\varphi_1 \geq \varphi_2$ in $B_0(\Sigma, K)$. Since I is normalized, $\min_{J \in \Phi_\varphi} J(k) = k$ for all $k \in K$.

Now we prove that (i) implies (iii). We start showing that (i) implies (ii). By Lemma 6 we have that I admits a positively superhomogeneous, and monotone extension \tilde{I} on $B_0(\Sigma, K_\infty)$. Moreover, since I is continuous, and K is either upper open or an order interval, Lemma 7 yields that \tilde{I}^+ is an upper semicontinuous, positively superhomogeneous, and monotone normalized extension of I . Let $\varphi \in B_0(\Sigma, K_\infty)$. If $\varphi(s) = 0$ for all $s \in S$, we let $\Psi_\varphi = \{\mathbf{0}\}$ and, since $0 \in K$, by the normalization of I we have that $I(\varphi) = \max_{T \in \Psi_\varphi} T(\varphi) = 0$. Clearly $\mathbf{0}$ is monotone and superlinear. Now suppose that $\varphi(s) > 0$ for some $s \in S$. Let $\xi \in B_0(\Sigma, K_\infty)$ with $\varphi \geq \xi$ and $\xi(s) > 0$ for some $s \in S$. We have that for all $s' \in S$,

$$\varphi(s') \geq \xi(s') \inf_{s: \xi(s) > 0} \left\{ \frac{\varphi(s)}{\xi(s)} \right\}.$$

Indeed, if $\xi(s') = 0$, then the inequality follows as $\varphi(s') \geq 0$. If $\xi(s') > 0$, then

$$\varphi(s') = \xi(s') \frac{\varphi(s')}{\xi(s')} \geq \xi(s') \inf_{s:\xi(s)>0} \left\{ \frac{\varphi(s)}{\xi(s)} \right\}.$$

Paired with monotonicity and positive superhomogeneity of \tilde{I}^+ , this inequality yields the following

$$\tilde{I}^+(\varphi) \geq \max_{\xi \neq \mathbf{0}, \varphi \geq \xi} \tilde{I}^+ \left(\xi \inf_{s:\xi(s)>0} \left\{ \frac{\varphi(s)}{\xi(s)} \right\} \right) \geq \max_{\xi \neq \mathbf{0}, \varphi \geq \xi} \tilde{I}^+(\xi) \inf_{s:\xi(s)>0} \left\{ \frac{\varphi(s)}{\xi(s)} \right\} \geq \tilde{I}^+(\varphi)$$

where the second-to-last inequality follows from $\inf_{s:\xi(s)>0} \{\varphi(s)/\xi(s)\} \geq 1$, for all non-identically zero $\xi \in B_0(\Sigma, K_\infty)$ with $\varphi \geq \xi$. For all non-identically zero $\xi \in B_0(\Sigma, K_\infty)$, define $P_\xi(\psi) = I(\xi) \inf_{s:\xi(s)>0} \{\psi(s)/\xi(s)\}$ for all $\psi \in B_0(\Sigma, K_\infty)$. Since for all $\xi \in B_0(\Sigma, K_\infty)$, $\tilde{I}^+(\xi) \geq \tilde{I}^+(\mathbf{0}) = I(\mathbf{0}) = 0$, each P_ξ is monotone and superlinear. Therefore, letting $\Psi_\varphi = \{P_\xi : \xi \in B_0(\Sigma, K_\infty), \varphi \geq \xi, \xi \neq \mathbf{0}\}$ for all $\varphi \in B_0(\Sigma, K)$, we have found a family of monotone and superlinear functionals such that

$$I(\varphi) = \tilde{I}^+(\varphi) = \max_{H \in \Psi_\varphi} H(\varphi)$$

for all $\varphi \in B_0(\Sigma, K)$. Clearly, $\Psi_{\varphi_1} \subseteq \Psi_{\varphi_2}$ for all $\varphi_1 \leq \varphi_2$ in $B_0(\Sigma, K)$. Since I is normalized, $\max_{H \in \Psi_\varphi} H(k) = k$ for all $k \in K$.

To conclude we prove that (ii) implies (i), and that (iii) implies (i). Suppose that (ii) holds. It is immediate to notice that I must be monotone and normalized. Positive superhomogeneity follows from the fact that for all $\varphi \in B_0(\Sigma, K)$ and $\alpha \geq 1$, we have $\Phi_{\alpha\varphi} \subseteq \Phi_\varphi$. Indeed, these inclusions imply that

$$I(\alpha\varphi) = \min_{J \in \Phi_{\alpha\varphi}} J(\alpha\varphi) = \alpha \min_{J \in \Phi_{\alpha\varphi}} J(\varphi) \geq \alpha \min_{J \in \Phi_\varphi} J(\varphi) = \alpha I(\varphi).$$

The proof of (iii) implies (i) is analogous upon observing that $\Psi_\varphi \subseteq \Psi_{\alpha\varphi}$. □

PROPOSITION 18. *Fix a convex $K \subseteq \mathbb{R}$ with $\min K = 0$ and a continuous map $I : B_0(\Sigma, K) \rightarrow \mathbb{R}$. Then, the following are equivalent*

- (i) *I is monotone, normalized, and positively superhomogeneous.*
- (ii) *For all $\varphi \in B_0(\Sigma, K)$, there exists a set D_φ of upper semicontinuous and quasiconcave*

$d : \Delta(S) \rightarrow [0, \infty]$,

$$I(\varphi) = \max_{d \in D_\varphi} \inf_{p \in \Delta(S)} \frac{\int_S \varphi dp}{d(p)}.$$

Moreover, $\max_{d \in D_k} \inf_{p \in \Delta(S)} k/d(p) = k$ for all $k \in K$ and $D_{\varphi_1} \subseteq D_{\varphi_2}$ for all $\varphi_2 \geq \varphi_1$ in $B_0(\Sigma, K)$.

Proof. By Proposition 17, for all $\varphi \in B_0(\Sigma, K)$, there exists a family Ψ_φ of monotone, superlinear functionals, mapping from $B_0(\Sigma, K)$ to \mathbb{R} , such that

$$I(\varphi) = \max_{H \in \Psi_\varphi} H(\varphi).$$

Moreover, $\max_{H \in \Psi_\varphi} H(k) = k$ for all $k \in K$, and $\Psi_{\varphi_1} \subseteq \Psi_{\varphi_2}$ for all $\varphi_2 \geq \varphi_1$ in $B_0(\Sigma, K)$. Fix $\varphi \in B_0(\Sigma, K)$. Each $H \in \Psi_\varphi$ is real-valued, and, being monotone and superlinear, also continuous, and hence, by Theorem 36 in Cerreia-Vioglio et al. (2011c),

$$H(\psi) = \inf_{p \in \Delta(S)} G_H \left(\int_S \psi dp, p \right)$$

for all $\psi \in B_0(\Sigma, K)$. By positive homogeneity of H , Lemma 8 yields that G_H is also positively homogeneous in the first entry, and hence

$$H(\psi) = \inf_{p \in \Delta(S)} \int_S \psi dp G_H(1, p)$$

for all $\psi \in B_0(\Sigma, K)$. Given that H is real-valued, it follows that there must exist $p \in \Delta(S)$ such that $G_H(1, p) < \infty$. Define $d_H : \Delta(S) \rightarrow [0, \infty]$ as $d_H(p) = 1/G_H(1, p)$. By the previous observation, there must exist $p \in \Delta(S)$ such that $d_H(p) > 0$. Since $H \in \Psi_\varphi$ is monotone, superlinear, by Lemmas 31 and 32 in Cerreia-Vioglio et al. (2011c), G_H is quasiconvex and lower semicontinuous.²⁷ It follows that, for all $t \in \mathbb{R}$,

$$\{p \in \Delta(S) : d_H(p) \leq t\} = \{p \in \Delta(S) : G_H(1, p) \geq 1/t\}$$

are closed and convex for all $t \geq 0$. Thus, d_H is quasiconcave and upper semicontinuous.

²⁷This claim is slightly more subtle, indeed notice that $B_0(\Sigma, K)$ here is not lower open. Though, since each H is monotone and superlinear, it admits a monotone, lower semicontinuous, and positively homogeneous extension, say \bar{H} , over the whole $B_0(\Sigma)$. The G_H we consider here should actually be $G_{\bar{H}}$ which, by the continuity of \bar{H} , is lower semicontinuous by Lemma 32 in Cerreia-Vioglio et al. (2011c).

Letting $D_\varphi = \{d_H : H \in \Psi_\varphi\}$, we have that

$$I(\varphi) = \max_{d \in D_\varphi} \inf_{p \in \Delta(S)} \frac{\int_S \varphi dp}{d(p)}.$$

By the normalization of I we have $\max_{d \in D_k} \min_{p \in \Delta(S)} k/d(p) = k$ for all $k \in K$. Moreover, $D_{\varphi_1} \subseteq D_{\varphi_2}$ for all $\varphi_2 \geq \varphi_1$ in $B_0(\Sigma, K)$, because for all such $\varphi_1, \varphi_2 \in B_0(\Sigma, K)$, we have $\Psi_{\varphi_1} \subseteq \Psi_{\varphi_2}$. The converse is proved using the same exact steps of (iii) implies (i) in Proposition 17. \square

B Proofs of the results in the main text

Proof of Lemma 1. The if part is straightforward. Therefore, we prove the only if part. The existence of an affine u follows from Theorem 8 in Herstein and Milnor (1953). Moreover, by Proposition 1 in Cerreia-Vioglio et al. (2011a) there exists a continuous, monotone, and normalized functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that $I \circ u$ represents \succsim . The rest of the claim follows from Proposition 10. \square

B.1 Absolute ambiguity attitudes

The proofs of the results concerning absolute ambiguity attitudes are based on the following result which generalizes Lemma 3 in Xue (2020). For any subset $K \subseteq \mathbb{R}$, denote by $\text{int}K$ the interior of K .

LEMMA 9. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) \succsim is MBA, and exhibits decreasing absolute ambiguity aversion.
- (ii) *There exist an affine function $u : X \rightarrow \mathbb{R}$ and a continuous, monotone, normalized, and constant superadditive functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,*

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Proof. The if part is straightforward. We prove the only if part. In particular, the existence of an affine u follows from Theorem 8 in Herstein and Milnor (1953). Since u is unique up to positive affine transformations, it is without loss of generality to assume that $0 \in \text{int}u(X)$ and denote by $x_0 \in X$ the element such that $u(x_0) = 0$. Moreover, by Proposition 1 in Cerreia-Vioglio et al. (2011a) there exists a continuous, monotone, and normalized

functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that $I \circ u$ represents \succsim . We now show the constant superadditivity of I . Since $I \circ u$ represents \succsim , decreasing absolute ambiguity aversion yields

$$I(\alpha\varphi + (1 - \alpha)u(x_0)) \geq u(\alpha x + (1 - \alpha)x_0) \implies I(\alpha\varphi + (1 - \alpha)k) \geq \alpha u(x) + (1 - \alpha)k \quad (12)$$

for all $\alpha \in (0, 1)$, $k \in u(X)$ with $k \geq 0$, $\varphi \in B_0(\Sigma, u(X))$, and $x \in X$. Suppose that, for some $\alpha \in (0, 1)$, $k \in u(X) \cap \mathbb{R}_+$, and $\varphi \in B_0(\Sigma, u(X))$, we have that

$$I(\alpha\varphi + (1 - \alpha)k) < I(\alpha\varphi) + (1 - \alpha)k. \quad (13)$$

Notice that since I is monotone and normalized and $u(X)$ is convex, $I(\alpha\varphi) \in \alpha u(X)$ as $\alpha\varphi(s) \in \alpha u(X)$ for all $s \in S$. Therefore, there exists $x \in X$, such that $I(\alpha\varphi) = \alpha u(x)$ and, since u is affine, $I(\alpha\varphi) = u(\alpha x + (1 - \alpha)x_0)$. Then, by (12) and (13)

$$I(\alpha\varphi + (1 - \alpha)u(x_0)) < u(\alpha x + (1 - \alpha)x_0) = I(\alpha\varphi) = I(\alpha\varphi + (1 - \alpha)u(x_0))$$

a contradiction. Thus, by Lemma 3 in Appendix A, I is constant superadditive. \square

Proof of Theorem 1. The only if part follows by Lemma 9 and Proposition 12 in Appendix A. For the if part define $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ as

$$I(\varphi) = \max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G\left(\int \varphi dp, p\right)$$

for all $\varphi \in B_0(\Sigma, K)$. Since \mathcal{G} is linearly continuous, we have that I is continuous. Moreover, I is normalized as $\max_{G \in \mathcal{G}} \inf_{p \in \Delta(S)} G(k, p)$ for all $k \in u(X)$. Since each $G \in \mathcal{G}$ is monotone and constant superadditive in the first argument, we have that I is monotone and constant superadditive. Thus, by Lemma 9, \succsim is an MBA preference relation exhibiting decreasing absolute ambiguity aversion. \square

Proof of Proposition 1. It follows by Lemma 9 and Proposition 14 in Appendix A. \square

B.2 Relative ambiguity attitudes

LEMMA 10. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) \succsim is MBA, admits a worst consequence, and exhibits decreasing relative ambiguity aversion.

(ii) There exist an affine function $u : X \rightarrow \mathbb{R}$ with $0 = \min u(X)$, and a continuous, monotone, normalized, and positively superhomogeneous functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Proof. We start with the if part. Since $0 = \min u(X)$ and I is continuous, monotone, and normalized we have that \succsim is an MBA preference admitting a worst consequence, say x_* . Therefore, we prove that \succsim satisfies decreasing relative ambiguity aversion. To this end suppose that $\alpha f + (1 - \alpha)x_* \succsim \alpha y + (1 - \alpha)x_*$ for some $\alpha \in (0, 1]$, $f \in \mathcal{F}$, and $y \in X$. Then, since $u(x_*) = 0$, u is affine, and I is normalized we have $I(\alpha u(f)) \geq \alpha u(y)$. Since I is positively superhomogeneous, we have that $\gamma \mapsto I(\gamma u(f))/\gamma$ is increasing in $(0, 1]$, and hence for all $\beta \in [\alpha, 1]$ we have $I(\beta u(f)) \geq \beta u(y)$. This yields

$$I(u(\beta f + (1 - \beta)x_*)) = I(\beta u(f)) \geq \beta u(y) = u(\beta y + (1 - \beta)x_*)$$

and hence $\beta f + (1 - \beta)x_* \succsim \beta y + (1 - \beta)x_*$.

We now prove the only if part. Analogously to the proofs of Lemmas 1 and 9 there exists an affine u and a continuous, monotone, and normalized functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that $I \circ u$ represents \succsim . We are left to show the positively superhomogeneity of I . First of all, since $I \circ u$ represents \succsim , decreasing relative ambiguity aversion implies that

$$I(\alpha\varphi + (1 - \alpha)u(x_*)) \geq u(\alpha x + (1 - \alpha)x_*) \implies I(\beta\varphi + (1 - \beta)u(x_*)) \geq u(\beta x + (1 - \beta)x_*) \quad (14)$$

for all $\alpha, \beta \in (0, 1]$ with $\alpha \leq \beta$, $\varphi \in B_0(\Sigma, u(X))$, $x \in X$. By the affinity of u , (14) can be rewritten as

$$I(\alpha\varphi) \geq \alpha k \implies I(\beta\varphi) \geq \beta k$$

for all $\alpha, \beta \in (0, 1]$ with $\alpha \leq \beta$ and $k \in u(X)$. Since $I(B_0(\Sigma, u(X))) \subseteq u(X)$, it follows that $\gamma \mapsto I(\gamma\varphi)/\gamma$ is increasing in $(0, 1]$, and hence

$$I(\gamma\varphi) \leq \gamma I(\varphi)$$

for all $\gamma \in (0, 1]$ and $\varphi \in B_0(\Sigma, u(X))$, showing the statement. \square

Proof of Theorem 2. The only if part follows by Lemma 10 and Proposition 16 in Appendix

A. The if part is analogous to the proof of Theorem 1 substituting constant superadditivity with positive superhomogeneity. \square

Proof of Proposition 2. It follows by Lemma 10 and Proposition 18 in Appendix A. \square

B.3 Examples 1 and 3

We first report a fairly standard preliminary result. The points 1 and 2 below are due to Marinacci and Montrucchio (2010) (see Appendix C therein).

LEMMA 11. *If $K \subseteq \mathbb{R}$ is convex and $\phi : K \rightarrow \mathbb{R}$ is strictly increasing, continuous, concave, then $I_q : B_0(\Sigma, K) \rightarrow \mathbb{R}$ is continuous, normalized, monotone, and quasiconcave for $q \in Q$. Moreover*

- (i) *If $K = [0, \infty)$ and ϕ is twice differentiable in $(0, \infty)$ and DARA, then I_q is constant superadditive.*
- (ii) *If $K = [0, \infty)$ and ϕ is twice differentiable in $(0, \infty)$ and DRRA, then I_q is positively superhomogeneous.*

Proof. Since ϕ is strictly increasing, monotonicity and normalization are immediate. As per continuity, suppose that $\psi_n \rightarrow \psi$. Then, since ϕ is continuous we have that $\phi(\psi_n) \rightarrow \phi(\psi)$ pointwise, and by the monotonicity of ϕ , we have that $|\phi(\psi_n)| \leq \phi(\|\psi_n\|_\infty)$ for all $n \in \mathbb{N}$. Since $\|\psi_n\|_\infty \rightarrow \|\psi\|_\infty$, we have that there exists $K > 0$ such that $\sup_n \|\psi_n\|_\infty \leq K$. Thus, for all $n \in \mathbb{N}$, we have that $|\phi(\psi_n)| \leq \phi(K)$. Then, by the dominated convergence theorem, we have that

$$\int \phi(\psi_n) dq \rightarrow \int \phi(\psi) dq.$$

Since, ϕ^{-1} is also continuous, it follows that $I_q(\psi_n) \rightarrow I_q(\psi)$, proving the continuity. Moreover, since ϕ^{-1} is strictly increasing and ϕ is concave, we have that

$$\begin{aligned} I_q(\alpha\varphi + (1-\alpha)\psi) &= \phi^{-1} \left(\int \phi(\alpha\varphi + (1-\alpha)\psi) dq \right) \geq \phi^{-1} \left(\int \alpha\phi(\varphi) + (1-\alpha)\phi(\psi) dq \right) \\ &\geq \min \left\{ \phi^{-1} \left(\int \phi(\varphi) dq \right), \phi^{-1} \left(\int \phi(\psi) dq \right) \right\} = \min \{ I_q(\varphi), I_q(\psi) \} \end{aligned}$$

for all $\alpha \in [0, 1]$, and $\varphi, \psi \in B_0(\Sigma, K)$, where the last inequality follows from the fact that being monotone ϕ^{-1} is also quasiconcave.

Points (i) and (ii) follow from Marinacci and Montrucchio (2010) Theorem 12 and Corollary 1 (see also Proposition 6 therein), respectively. Their proofs are presented in a finite state space, though this is not necessary for the argument.²⁸ This concludes the proof. \square

We say that a function $G : \mathbb{R} \times \Delta(S) \rightarrow \bar{\mathbb{R}}$ is *linearly continuous* if the map

$$\varphi \mapsto \inf_{p \in \Delta(S)} G \left(\int \varphi dp, p \right)$$

is continuous.

LEMMA 12. *For all $q \in Q$, there exists a linearly continuous $G_q : \mathbb{R} \times \Delta(S) \rightarrow \bar{\mathbb{R}}$ monotone in the first entry, quasiconvex, and such that for all $\varphi \in B_0(\Sigma, K)$*

$$I_q(\varphi) = \inf_{p \in \Delta(S)} G_q \left(\int \varphi dp, p \right).$$

Moreover,

- (i) *If ϕ is DARA, then G_q is constant superadditive in the first argument.*
- (ii) *If ϕ is DRRA, then G_q is positively superhomogeneous in the first argument.*

Proof. By Lemma 11, we have that each I_q is continuous, normalized, monotone, and quasiconcave. Therefore, by Theorem 36 in Cerreia-Vioglio et al. (2011c), we have that for all $\varphi \in B_0(\Sigma, K)$,

$$I_q(\varphi) = \inf_{p \in \Delta(S)} G_q \left(\int \varphi dp, p \right)$$

where $G_q(t, p) = \sup \left\{ I_q(\varphi) : \varphi \in B_0(\Sigma, K) \text{ and } \int \varphi dp \leq t \right\}$ for all $(t, p) \in \mathbb{R} \times \Delta(S)$. Following the results in Cerreia-Vioglio et al. (2011c) we have that G_q is monotone in the first entry, quasiconvex, and such that for all $k \in K$, $\inf_{p \in \Delta(S)} G_q(k, p) = k$. Since I_q is continuous, we have that G_q is linearly continuous. Moreover,

- (i) If ϕ is DARA, by Lemma 11, we have that I_q is constant superadditive, and hence by Proposition 8, we have that G_q is constant superadditive.
- (ii) If ϕ is DRRA, by Lemma 11, we have that I_q is positively superhomogeneous, and hence by Proposition 9, we have that G_q is positively superhomogeneous.

²⁸The details are available upon request.

These conclude the proof. □

By Lemma 11 and 12, MBA preferences \succsim represented by $f \mapsto \max_{q \in Q} I_q(u(f))$ satisfy: (i) if ϕ is DARA, then \succsim satisfies the conditions of Theorem 1; (ii) if ϕ is DRRA, then \succsim satisfies the conditions of Theorem 2. We provide two examples of functions $\phi : [0, \infty) \rightarrow \mathbb{R}$ that satisfy these hypotheses:

- $\phi(t) = \sqrt{t}$ for all $t \geq 0$, is strictly increasing, continuous, concave, and DARA. To see the latter, notice that

$$-\frac{\phi''(t)}{\phi'(t)} = \frac{1}{2t},$$

a (strictly) decreasing function.

- $\phi(t) = \sqrt{t} + t$ for all $t \geq 0$, is strictly increasing, continuous, concave, and DRRA. To see the latter, notice that, for all $t > 0$,

$$-t \frac{\phi''(t)}{\phi'(t)} = \frac{1}{4\sqrt{t} + 2}$$

which is decreasing in t .

B.4 Remark 2

PROPOSITION 19. *Let $\phi : [1, \infty) \rightarrow \mathbb{R}$ strictly increasing, twice differentiable on $(1, \infty)$, and concave and suppose that Σ is non-trivial. The functional $I_{\phi, \mu} : B_0(\Sigma, [1, \infty)) \rightarrow \mathbb{R}$ defined as*

$$I_{\phi, \mu}(\varphi) = \phi^{-1} \left(\int \phi \left(\int \varphi dp \right) d\mu \right)$$

for all $\varphi \in B_0(\Sigma, [1, \infty))$ is positively superhomogeneous for all countably additive probabilities μ over $\Delta^\sigma(S)$ if and only if ϕ is DRRA.

Proof. Fix a countable additive μ over $\Delta^\sigma(S)$. First we prove that $I_{\phi, \mu}$ is positively superhomogeneous if and only if $\hat{I}_{\phi, \mu}$ defined as

$$\hat{I}_{\phi, \mu} : \varphi \mapsto \log \left(\phi^{-1} \left(\int \phi \left(\int e^\varphi dp \right) d\mu \right) \right)$$

is constant superadditive. First suppose that $I_{\phi, \mu}$ is positively superhomogeneous. For all

$\varphi \in B_0(\Sigma, [0, \infty))$ and $k \geq 0$, we have that

$$\begin{aligned}\hat{I}_{\phi, \mu}(\varphi + k) &= \log\left(\phi^{-1}\left(\int \phi\left(\int e^{\varphi+k} dp\right) d\mu\right)\right) = \log\left(\phi^{-1}\left(\int \phi\left(\int e^{\varphi} e^k dp\right) d\mu\right)\right) \\ &\geq \log\left(e^k \phi^{-1}\left(\int \phi\left(\int e^{\varphi} dp\right) d\mu\right)\right) = \log\left(\phi^{-1}\left(\int \phi\left(\int e^{\varphi} dp\right) d\mu\right)\right) + k = \hat{I}_{\phi, \mu}(\varphi) + k.\end{aligned}$$

Therefore, $\hat{I}_{\phi, \mu}$ is constant superadditive. Conversely, suppose that $\hat{I}_{\phi, \mu}$ is constant superadditive. Then, we have that for all $\alpha \geq 1$ and $\varphi \in B_0(\Sigma, [1, \infty))$,

$$I_{\phi, \mu}(\alpha\varphi) = e^{\hat{I}_{\phi, \mu}(\log(\alpha\varphi))} = e^{\hat{I}_{\phi, \mu}(\log(\alpha) + \log(\varphi))} \geq e^{\hat{I}_{\phi, \mu}(\log(\varphi))} e^{\log(\alpha)} = \alpha e^{\hat{I}_{\phi, \mu}(\log(\varphi))} = \alpha I_{\phi, \mu}(\varphi)$$

Therefore, $I_{\phi, \mu}$ is positively superhomogeneous. In particular, notice that $\hat{I}_{\phi, \mu} = I_{\hat{\phi}, \mu}$ with $\hat{\phi}(t) = \phi(e^t)$ for all $t \geq 0$. Notice that

$$\hat{\phi}'(t) = e^t \phi'(e^t) \text{ and } \hat{\phi}''(t) = e^t \phi'(e^t) + e^{2t} \phi''(e^t),$$

which implies

$$\hat{\phi}''(t) \leq 0 \Leftrightarrow \phi'(e^t) \leq -e^t \phi''(e^t) \Leftrightarrow 0 \leq -e^t \frac{\phi''(e^t)}{\phi'(e^t)} \Leftrightarrow 0 \leq -\frac{\phi''(e^t)}{\phi'(e^t)}$$

Since ϕ is strictly increasing, we have $\phi'(e^t) > 0$; since ϕ is concave, we have $\phi''(e^t) \leq 0$, and hence $\hat{\phi}''(t) \leq 0$ for all t . Therefore, $\hat{\phi}$ is strictly increasing and concave. Moreover, by Proposition 7 in Xue (2020),²⁹ $\hat{I}_{\phi, \mu}$ is constant superadditive for all countable additive probabilities μ in $\Delta^\sigma(S)$ if and only if $\hat{\phi}$ is DARA. In turn, $\hat{\phi}$ is DARA if and only if ϕ is DRRA. Therefore, connecting all the proved equivalences, $I_{\phi, \mu}$ is positively superhomogeneous if and only if ϕ is DRRA. \square

B.5 Risk sharing

LEMMA 13. *Let \succsim be a binary relation over \mathcal{F} . The following are equivalent*

- (i) *\succsim is MBA, admits a worst consequence, and exhibits decreasing absolute ambiguity aversion and increasing relative ambiguity aversion.*

²⁹Notice that in the proof Xue (2020) uses the full unboundedness requirement only for the preferential characterization of constant superadditivity in terms of decreasing absolute ambiguity aversion. The rest of the proof, and in particular the cited results from Cerreia-Vioglio et al. (2011c), do not depend on the full unboundedness condition.

(ii) There exist an affine function $u : X \rightarrow \mathbb{R}$ with $\min u(X) = 0$ and a continuous, monotone, normalized, constant superadditive, and positively subhomogeneous functional $I : B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that, for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)).$$

Moreover, the same statement holds if we require \succsim to satisfy non-constant decreasing absolute ambiguity aversion and non-constant increasing relative ambiguity aversion, along with strict constant superadditivity and strict positively subhomogeneity.

Proof. The equivalence between the two points follows immediately by combining Lemma 9 and Lemma 10, adapting the latter specularly to increasing relative ambiguity aversion. Therefore, we only need to prove that non-constant decreasing absolute ambiguity aversion is equivalent to the strict constant superadditivity of I , and the analogous statement for non-constant increasing relative ambiguity aversion and the strict positive subhomogeneity of I . The proofs of such equivalences are omitted for the sake of brevity as they are completely analogous (up to changing few weak signs with their strict counterparts) to the proofs of the lemmas mentioned above.³⁰ \square

Proof of Proposition 3. Let $x \in \mathbb{R}_{++}$ and $g \neq x$ in \mathbb{R}_+^S with $V(g) \geq V(x)$. Then, for all $\lambda \in [0, 1]$ it holds that

$$V(\lambda g + (1 - \lambda)x) \geq V(\lambda g) + (1 - \lambda)V(x) \geq \lambda V(g) + (1 - \lambda)V(x) \geq \lambda V(x) + (1 - \lambda)V(x) = V(x),$$

with at least one of the inequalities being strict by assumption. Since V is nice, the result follows by Proposition 2 (points (2) and (3)) in Ghirardato and Siniscalchi (2018b). \square

Proof of Corollary 1. The result follows by a direct application of Theorem 3 by Ghirardato and Siniscalchi (2018b) and Proposition 3. \square

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³⁰The details are available upon request.

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