

Recursive Preferences and Ambiguity Attitudes

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Abstract

We illustrate the strong implications of recursivity, a standard assumption in dynamic environments, on attitudes toward uncertainty. We show that in intertemporal consumption choice problems, recursivity always implies constant absolute ambiguity aversion (CAAA) when applying the standard dynamic extension of monotonicity. Our analysis yields a functional equation called “generalized rectangularity”, as it generalizes the standard notion of rectangularity for recursive multiple priors. Our results highlight that if uncertainty aversion is modeled as a form of convexity, recursivity limits us to only recursive variational preferences. We propose a novel notion of monotonicity that enables us to overcome this limitation.

Keywords: Dynamic choice, recursive utility, uncertainty aversion, absolute attitudes, generalized rectangularity.

JEL classification: C61, D81.

1 Introduction

Recursive preferences are a key tool for dynamic economic models. They are the workhorse in macroeconomics and finance for studying a variety of different problems,

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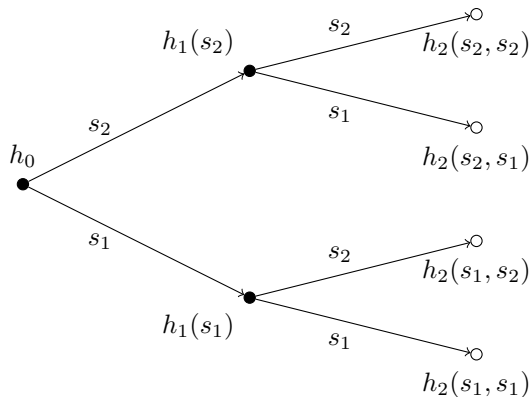


Figure 1: Illustration of a consumption program.

ranging from consumption-based asset pricing (Epstein and Zin (1989), Epstein and Zin (1991)), precautionary savings (Weil (1989), Hansen et al. (1999)), business cycles (Tallarini (2000), Anderson (2005)), and climate change (Cai and Lontzek (2019)). In dynamic models with strategic interaction, they have recently been used to study repeated games (Kochov and Song (2021)) and Bayesian Persuasion (Pahlke (2022)).

Recursivity entails several restrictions on dynamic choice behavior, among which one of the key assumptions is a notion of time or dynamic consistency, i.e., that at every time period the decision maker will carry out the plan of actions that was determined to be optimal ex-ante. The assumption of recursivity provides analytical tractability in that it permits the use of well-known tools from dynamic programming.

In this paper, we show that recursivity has strong restrictions on attitudes toward uncertainty, i.e., how uncertainty attitudes change when individuals become better off overall. We focus on a major class of dynamic choice problems, which we refer to as intertemporal consumption choice problems. These problems take place over long horizons, and the source of utility is a consumption stream. Figure 1 offers a graphical representation of a (stochastic) consumption stream. At every period, a shock $s \in S$ is realized, and the total sequence of shocks determines the consumption level at any given time period.

For such problems the implications of recursivity for ambiguity attitudes depend on the notion of monotonicity that is employed. Under the standard notion of monotonicity adopted in the literature (see e.g. Epstein and Schneider (2003b) and Maccheroni et al. (2006b)), our first result (Theorem 2 and Corollary 1) shows that recursive preferences *always* satisfy constant absolute ambiguity aversion (CAAA).

As a byproduct, we obtain a generalized “rectangularity” condition for recursive preferences that satisfy this notion of recursivity. Similarly to rectangularity from Epstein and Schneider (2003a, 2003b), generalized rectangularity characterizes “generalized beliefs”—as modeled by certainty equivalents—that are dynamically consistent. Our generalized “rectangularity” condition for an ex-ante and one-step-ahead certainty equivalent can be written as:

$$I_0(\xi) = \beta I_{+1} \left(\frac{1}{\beta} I_0(\xi^1) \right), \quad (1)$$

for every random variable ξ , where I_0 and I_{+1} are certainty equivalents reflecting ex-ante and one-step-ahead beliefs, ξ^1 denotes a shift operator applied to the random variable ξ , and $\beta \in (0, 1)$ is a discount factor.

Condition (1) can be seen as a generalized law of iterated expectations in a dynamic setting with intertemporal consumption. The condition has a simple interpretation. Beliefs over an entire sequence of states can be decomposed into beliefs over the next-period state and conditional beliefs over the entire sequence of states. The above functional equation highlights that in order to elicit the ex-ante certainty equivalent I_0 —which can be interpreted as describing beliefs about the entire realizations of shocks—it is enough to elicit the one-step-ahead certainty equivalent. Our Theorem 3 provides the formalization of the previous observation: given the one-step ahead certainty equivalent I_{+1} generalized rectangularity allows the analyst to recover I_0 .¹

In the special case of recursive maxmin expected utility (MEU) preferences condition (1) is shown to be equivalent to the sets of ex-ante beliefs \mathcal{P} and one-step-ahead beliefs \mathcal{L} be connected by the relationship

$$\min_{P \in \mathcal{P}} P(A) = \min_{\ell \in \mathcal{L}} \left[\sum_{s \in S} \ell(s) \min_{P \in \mathcal{P}} P(A_s) \right]$$

which is equivalent to the notion of rectangularity introduced by Epstein and Schneider (2003a).²

Another important implication is that if one assumes that monotone preferences, on top of recursivity, satisfy the notion of uncertainty aversion introduced by Gilboa

¹If the functional equation does not admit an analytic solution, I_0 can be determined by means of the numerical methods we develop.

²See Equation 2.4 in their paper.

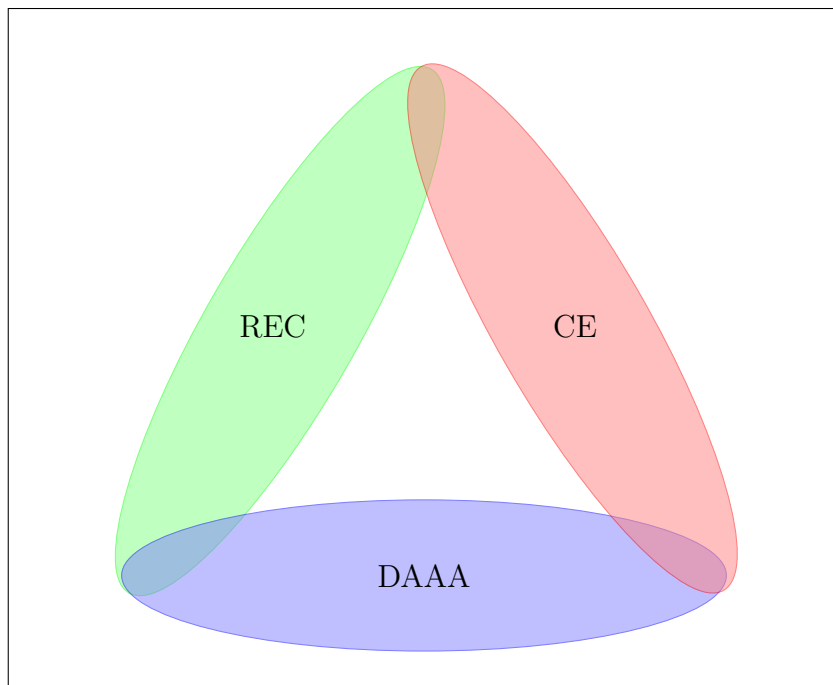


Figure 2: A decision-theoretic trilemma: no preferences satisfy recursivity (REC), non-trivial decreasing absolute ambiguity aversion (DAAA) and admit a representation by an ex-ante certainty equivalent (CE).

and Schmeidler (1989) we have that preferences must admit a variational representation in the sense of Maccheroni et al. (2006a). More specifically, following Cerreia-Vioglio et al. (2011), suppose that there are a utility index u and a quasiconvex function G such that DM's preferences admit the recursive representation

$$V(h) = u(h_0) + \beta \inf_{\ell \in \Delta(S)} G \left(\sum_{s \in S} \ell(s) [V \circ h^{s,1}], \ell \right).$$

If such preferences satisfy state-monotonicity, our results imply that the aggregator G has the form

$$G(t, \ell) = t + c(\ell),$$

where $c : \Delta(S) \rightarrow [0, \infty]$ is a convex function.

The assumption that a decision maker's attitudes towards uncertainty remain unchanged as they become better off contrasts with both introspection and existing experimental literature (see for example Baillon and Placido (2019)).³ Our results

³As stated by Baillon and Placido (2019) at p. 325: "Our findings seem to encourage the use

therefore suggest, broadly speaking, the following modeling trade-off: one has to choose between tractability—as reflected by recursivity (REC), experimental validity, represented by non-trivial decreasing absolute ambiguity aversion (DAAA), and preferences that admit a representation by means of an ex-ante certainty equivalent (CE). This trilemma is depicted in Figure 2.

Because of this problem, for intertemporal consumption choice problems we propose a different notion of monotonicity, which we refer to as state-time monotonicity (Axiom I.2). This notion is a basic consistency principle which requires that an uncertain consumption plan is preferred to another, whenever such a ranking holds jointly at any possible state of the world and at any possible time period. State-time monotonicity allows us to keep DAAA and REC together while dropping CE.

Finally, building on Epstein (1992), we also consider a different major formalization of a dynamic choice problem: sequential choice problems. Sequential choice problems take place over short intervals of time during which consumption plans can be taken to be fixed, and the source of utility is terminal wealth as opposed to the latter case in which it is given by a consumption stream. Sequential choice problems are typically employed when dealing with updating rules (see e.g., Pires (2002), Klibanoff and Hanany (2007)).

In this setting, the implications of recursivity are more nuanced. Savochkin et al. (2022), *inter alia*, show that CAAA is implied by recursivity when preferences admit a smooth ambiguity representation. We show that in general recursivity imposes no restriction on uncertainty attitudes for sequential choice problems. We provide an example of recursive preferences that can allow for unrestricted uncertainty attitudes. However, our examples feature probabilistic sophistication, leaving the door open as to whether the result holds for more general classes of preferences that are not probabilistically sophisticated.

1.1 Organization of the paper

Section 2 introduces the notation and the main choice-theoretic objects used in the paper. The implications for intertemporal consumption choice problems in terms of ambiguity attitudes are studied in Section 2.3. Section 4.3 discusses the case of

of ambiguity models that are flexible enough to accommodate changes in ambiguity attitudes at increased utility levels.”

sequential choice problems. Section 5 concludes with a discussion of the results in light of the existing literature. All the proofs can be found in the Appendix.

2 Framework

2.1 Static choice problems and mathematical preliminaries

Let Ω be a nonempty set of *states of the world* and \mathcal{G} an algebra of *events* over it. By X we denote a convex subset of a vector space, interpreted as a set of *consequences*. A function $f : \Omega \rightarrow X$ is said to be a (simple) *act* if it is \mathcal{G} -measurable and $f(\Omega)$ is finite; the set of acts is denoted by \mathbf{F} . As usual we identify X as a subset of \mathbf{F} . We denote by \succsim a binary relation over \mathbf{F} , by \sim and \succ its symmetric and asymmetric parts, respectively. A function $V : \mathbf{F} \rightarrow \mathbb{R}$ represents \succsim if

$$f \succsim g \iff V(f) \geq V(g)$$

for all $f, g \in \mathbf{F}$. Fix $K \subseteq \mathbb{R}$, we denote by $B_0(K, \Omega, \mathcal{G})$ the set of bounded simple \mathcal{G} -measurable functions taking values in K , and denote by $B(K, \Omega, \mathcal{G})$ its closure in the supremum norm $\|\cdot\|_\infty$ (i.e., the space of bounded \mathcal{G} -measurable functions). We set $B_0(\Omega, \mathcal{G}) := B_0(\mathbb{R}, \Omega, \mathcal{G})$ and $B(\Omega, \mathcal{G}) := B(\mathbb{R}, \Omega, \mathcal{G})$. For all $A \in \mathcal{G}$, we denote by $\mathbf{1}_A$ its indicator function. Fix $K \subseteq \mathbb{R}$, a functional $I : B(K, \Omega, \mathcal{G}) \rightarrow \mathbb{R}$ is said to be *continuous* if $\lim_{n \rightarrow \infty} I(\xi_n) = I(\xi)$ for every sequence $(\xi_n)_{n=0}^\infty$ converging pointwise to ξ , and *normalized* if $I(k\mathbf{1}_\Omega) = k$ for all $k \in K$. We say that I is *monotone* if $I(\xi) \geq I(\xi')$ whenever $\xi \geq \xi'$, for all $\xi, \xi' \in B(K, \Omega, \mathcal{G})$. If I is continuous, monotone, and normalized, then it is said to be a *certainty equivalent*. Moreover, we say that I is *translation invariant* if $I(\xi + k\mathbf{1}_\Omega) = I(\xi) + k$, for all $\xi \in B(K, \Omega, \mathcal{G})$ and all $k \in K$ such that $\xi + k \in B(K, \Omega, \mathcal{G})$.

Given a nonempty set Y and an algebra \mathcal{A} over it, we denote by $\Delta(Y)$ the set of finitely additive probability measures over it, and $\Delta^\sigma(Y)$ the set of countably additive probabilities. For any nonempty set A , we will denote by $A^\infty := \prod_{t=1}^\infty A^t$ its countably infinite Cartesian product and by 2^A its power set.

2.1.1 Ambiguity attitudes

In the next sections, we will focus on dynamic choice problems. Building upon the work of Bommier et al. (2017) we will show how the combination of the classical

notions of *monotonicity* and *stationarity* of preferences will impose restrictions on the decision maker’s attitudes toward uncertainty. Before going into the dynamic setting, we discuss the notion of *constant absolute ambiguity aversion* as introduced by Grant and Polak (2013).

Definition 1. *A binary relation \succsim on \mathbf{F} exhibits constant absolute ambiguity aversion (CAAA) if for all $f \in \mathbf{F}$, $x, y, z \in X$, and $\alpha \in (0, 1)$,*

$$\alpha f + (1 - \alpha)x \succsim \alpha y + (1 - \alpha)x \implies \alpha f + (1 - \alpha)z \succsim \alpha y + (1 - \alpha)z.$$

In words, constant absolute ambiguity aversion requires that whenever an uncertain alternative is preferred to a sure outcome, “adding” the same certain alternative to both does not invert the preference. Absolute ambiguity attitudes have been thoroughly studied by Xue (2020) and Cerreia-Vioglio et al. (2019), in terms of utility and wealth, respectively. In the context of rational preferences, constant absolute ambiguity aversion is the same as requiring the certainty equivalent being translation invariant.

Proposition 1. *Consider a binary relation \succsim on \mathbf{F} . Suppose there exist an affine function such that $u : X \rightarrow \mathbb{R}$ and a translation invariant certainty equivalent $I : B_0(u(X), \Omega, \mathcal{G}) \rightarrow \mathbb{R}$ such that $f \mapsto I(u(f))$ represents \succsim . The following are equivalent*

- (i) \succsim satisfies constant absolute ambiguity aversion.
- (ii) I is translation invariant.

Proof. The proof is routine and hence it is omitted; the reader can consult Xue (2020). □

This proposition highlights how the constant absolute ambiguity aversion of preferences is translated to the representing certainty equivalent. Many notable models of decision-making under uncertainty satisfy constant absolute ambiguity aversion. Among others, maxmin (Gilboa and Schmeidler (1989)), α -maxmin (Ghirardato et al. (2004)), and variational (Maccheroni et al. (2006a)) models exhibit constant absolute ambiguity aversion. In their experimental work Baillon and Placido (2019) provide experimental evidence calling for the use of ambiguity models that can accommodate decreasing aversion toward ambiguity, rather than constant absolute ambiguity aversion.

2.2 Intertemporal consumption choice problems

Our focus will be on dynamic choice problems. Here, we formally present the setting we adopt which is analogous to Strzalecki (2013), but with an infinite horizon (see also Section A.1 of Bommier et al. (2017)). More specifically, we will adopt a stationary IID ambiguity setting as the one introduced by Epstein and Schneider (2003a). Let S be a finite set representing the states of the world to be realized in each period.⁴ We assume that S has at least three elements and let $\Sigma := 2^S$ be the associated algebra of events. The full state space is denoted by $\Omega := S^\infty$, with a state $\omega \in \Omega$ specifying a complete history (s_1, s_2, \dots) . In each period $t > 0$, the individual knows the partial history $s^t := (s_1, \dots, s_t)$. The evolution of such information can be represented by a filtration $(\mathcal{G}_t)_{t=0}^\infty$ on Ω where $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and $\mathcal{G}_t := \Sigma^t \times \{\emptyset, S\}^\infty$ for all $t > 0$. Let $\mathcal{G} = \sigma(\bigcup_{t=0}^\infty \mathcal{G}_t)$ for all $t > 0$, that is, the smallest sigma-algebra generated by the union of the sigma-algebras of the filtration $(\mathcal{G}_t)_{t=0}^\infty$. The relevant measurable space is (Ω, \mathcal{G}) .

The set of possible consumption levels is $[\underline{c}, \bar{c}]$ (with generic elements denoted by c, c', \hat{c}) with $\underline{c} < \bar{c}$. The entire consumption set is given by the set of lotteries over, that is $C = \Delta([\underline{c}, \bar{c}])$ (with generic elements denoted by x, y, z). We identify $[\underline{c}, \bar{c}]$ as a subset of C , looking at its elements as degenerate lotteries. A *consumption plan* is a C -valued, $(\mathcal{G}_t)_{t=0}^\infty$ -adapted stochastic process, that is, a sequence $h = (h_t)_{t=0}^\infty$ such that $h_t : \Omega \rightarrow C$ is \mathcal{G}_t -measurable for all $t \geq 0$. The set of all consumption plans is denoted by \mathbf{H} and it is endowed with the product topology (i.e., topology of pointwise convergence). We denote by $\mathbf{D} := C^\infty$ the set of all deterministic consumption plans. We identify C as a subset of \mathbf{D} where each $x \in C$ is seen as the constant consumption plan that yields the lottery x in each period. For all consumption plans $h \in \mathbf{H}$ and $s \in S$ define the *conditional consumption plan* $h^s \in \mathbf{H}$ by

$$h^s(s_1, s_2, \dots) = h(s, s_2, \dots) = (h_0, h_1(s, s_2, \dots), \dots)$$

for all $(s_1, s_2, \dots) \in \Omega$. In words, given a consumption plan $h \in \mathbf{H}$ and a state $s \in S$, the conditional consumption plan h^s is the consumption plan obtained from h when the decision maker knows that in the first period s realized. Now we can exploit conditional consumption plans to define the continuation of consumption plans. In particular, let $h = (h_0, h_1, h_2, \dots) \in \mathbf{H}$ and $s \in S$, the *continuation of h* , denoted by

⁴In finance and macroeconomics S is also interpreted as a set of *shocks*.

$h^{s,1}$, is defined as

$$h^{s,1}(s_1, s_2, \dots) = (h_1(s, s_2, \dots), h_2(s, s_2, \dots), \dots)$$

for all $(s_1, s_2, \dots) \in \Omega$. The continuation act $h^{s,1}$ is the consumption plan forwarding h by one period and knowing that in the first period state $s \in S$ realized. For all lotteries $x \in C$ and consumption plans $h \in \mathbf{H}$ we denote the concatenation (x, h) as,

$$(x, h)(s_1, s_2, \dots) = (x, h(s_2, s_3, \dots))$$

for all $(s_1, s_2, \dots) \in \Omega$. Likewise, let $s \in S$ and $\xi \in B(K, \Omega, \mathcal{G})$ where $K \subseteq \mathbb{R}$ is an interval, define $\xi^{s,1} \in B(K, \Omega, \mathcal{G})$ as

$$\xi^{s,1}(s_1, s_2, s_3, \dots) = \xi(s, s_2, s_3, \dots)$$

for all $(s_1, s_2, s_3, \dots) \in \Omega$. For all $h \in \mathbf{H}$, we will denote by h^1 the mapping $s \mapsto h^{s,1}$, ξ^1 is defined analogously.

To ease notation, since we will consider preferences that are dynamically consistent, here we only consider an ex-ante preference modeled by a binary relation \succsim on \mathbf{H} .⁵ Observe that in this setting the set of consumption plans \mathbf{H} can be seen as a subset of acts $\mathbf{F} \subseteq \mathbf{D}^\Omega$. Indeed an act here can be seen as a mapping from states into consumption streams

$$\omega \mapsto h(\omega) = (h_0, h_1(\omega), \dots) \in \mathbf{D}.$$

We study a product space and not a general filtration for several reasons. First, it is a standard setting in the decision-theoretic literature (see Strzalecki (2013) or Bommier et al. (2017)). Second, it is the natural setting to study attitudes toward uncertainty. With a general filtration, attitudes toward uncertainty will depend on changing beliefs.⁶

⁵Otherwise one would have to state all the axioms for the collection of preferences $(\succsim_{s^t})_{s^t}$ for every possible sequence $s^t \in S^t$, $t = 1, \dots$. Under the assumption of recursivity, it is not needed to consider this richer framework. More precisely, if preferences admit the recursive representation given in Definition 2 then it is possible to define conditional preferences $(\succsim_{s^t})_{s^t}$, $t = 1, \dots$ that satisfy the traditional notion of dynamic consistency.

⁶On this point, see the discussion in Strzalecki (2013) (pp. 1048-1049).

2.3 Recursive preferences

We consider now the intertemporal consumption choice setting described in Section 2.2. The primitive is a preference relation (i.e., a preorder) \succsim on the set of consumption plans \mathbf{H} . We consider preferences \succsim that admit a (separable) recursive representation as follows. Given $u : C \rightarrow \mathbb{R}$, we define $U : \mathbf{D} \rightarrow \mathbb{R}$ as

$$U(d) := \sum_{t=0}^{\infty} \beta^t u(d_t)$$

for all $d \in \mathbf{D}$.

Definition 2. A preference relation \succsim admits a (separable) recursive representation if there exists a tuple (V, I_{+1}, u, β) such that $V : \mathbf{H} \rightarrow \mathbb{R}$ represents \succsim and

$$V(h) = u(h_0) + \beta I_{+1}(V \circ h^1), \quad (2)$$

where $u : C \rightarrow \mathbb{R}$ is an affine utility function, $\beta \in (0, 1)$, $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \rightarrow \mathbb{R}$ is a certainty equivalent, and $V \circ h^1 : s \mapsto V(h^{s,1})$.

In the context of the previous definition, we refer to I_{+1} as a *one-step ahead certainty equivalent* and to \succsim as a *recursive preference relation*. The axiomatic characterization of recursive preferences is well understood in the literature (see for example de Castro and Galvao (2022), section 4 or Sarver (2018), Appendix A.1, for similar axiomatizations). Our contribution is to provide a novel axiomatization based on a weakening of the standard axiom of monotonicity used in the literature.

3 Axioms

Let \succsim represent the decision maker's preferences on \mathbf{H} . Next we state several properties (axioms) of the preference relation, which will be discussed and used to characterize recursive preferences.

Axiom I.1 (Weak order). \succsim is total and transitive.

We introduce the following notion of monotonicity.

Axiom I.2 (State-time monotonicity). For all $h, g \in \mathbf{H}$

$$[\forall \omega \in \Omega, t \geq 0, h_t(\omega) \succsim g_t(\omega)] \implies h \succsim g.$$

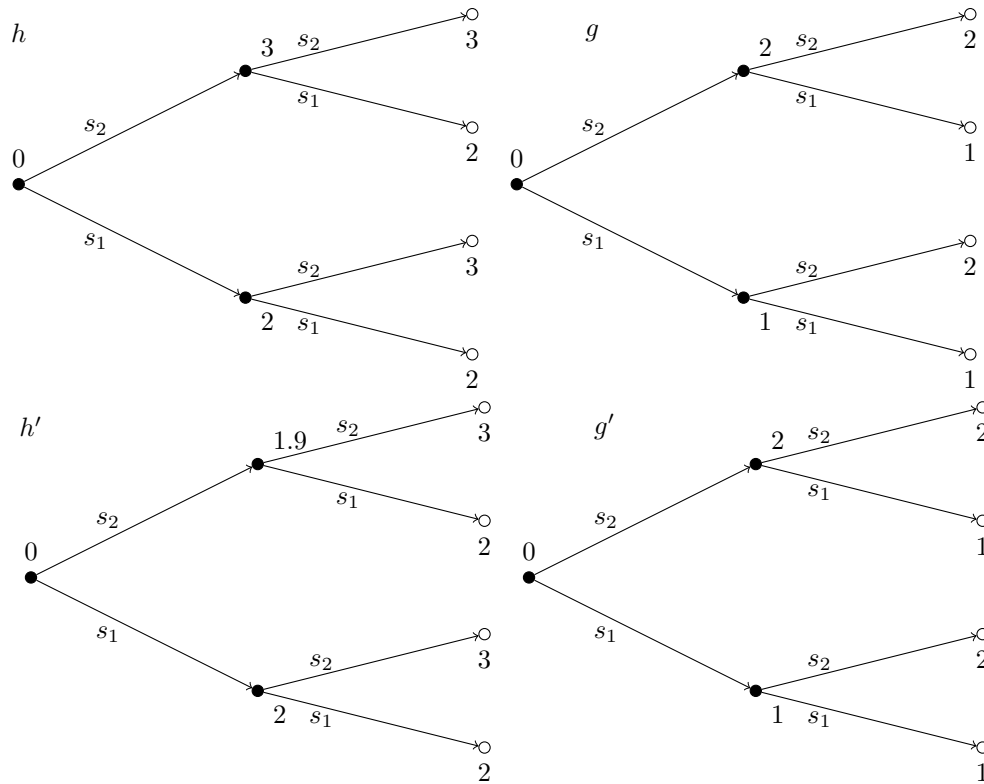


Figure 3: Illustration of state-time monotonicity

State-time monotonicity can be seen as a minimal consistency requirement. If a decision maker is asked to compare two uncertain consumption streams h and g and h weakly dominates g (according to the DM's preferences) in each period and each state, then h should be preferred to g .

A further interpretation of state-time monotonicity is to think of the *relevant* state space as being the set of all state-time pairs (ω, t) . Seeing each part of this combination as a node on our event tree, if the consumption level of h is higher than that of g , then h should be preferred to g . Figure 1 offers a graphical description of state-time monotonicity. The act h pays better than g at every possible node, while h' pays better than g' at every possible node except at the node s_1 . By state-time monotonicity h should be preferred to g but h' need not be preferred to g' . As we shall discuss, a stronger notion of monotonicity is typically adopted in the literature (e.g., see Epstein and Schneider (2003b), Maccheroni et al. (2006b), Bastianello and Faro (2022)).

Axiom I.3 (State-monotonicity). For all $h, g \in \mathbf{H}$,

$$[\forall \omega \in \Omega, (h_t(\omega))_{t=0}^\infty \succsim (g_t(\omega))_{t=0}^\infty] \implies h \succsim g.$$

State-monotonicity's interpretation is analogous to that of state-time monotonicity, but when the *relevant* state space is just Ω . One may expect state-monotonicity to be a stronger requirement than state-time monotonicity, this indeed the case when preferences over deterministic consumption streams are time separable. Bommier et al. (2017) show that state-monotonicity is equivalent to the translation invariance of the one-step ahead certainty equivalent I_{+1} . However, their characterization is silent about ambiguity attitudes. In our setting, their result can be seen as implying that \succsim has to satisfy constant absolute ambiguity aversion when restricted to one-step ahead acts.

The next two axioms are a technical and a monotonicity requirement on preferences.

Axiom I.4 (Continuity). For all $g \in \mathbf{H}$, the sets

$$\{h \in \mathbf{H} : h \succsim g\} \text{ and } \{h \in \mathbf{H} : g \succsim h\}$$

are closed with respect to the product topology.

Axiom I.5 (Monotonicity on levels of consumption). For all $c, \hat{c} \in [\underline{c}, \bar{c}]$ such that $c \geq \hat{c}$, we have $c \succsim \hat{c}$. Moreover, $\bar{c} \succ \underline{c}$.

Then, we will also consider two classical axioms in the literature on discounting and dynamic choice, namely *time separability* and *stationarity*.

Axiom I.6 (Time separability). For all $x, y, x', y' \in C$ and $d, d' \in \mathbf{D}$, $(x, y, d) \sim (x', y', d)$ if and only if $(x, y, d') \sim (x', y', d')$.

Consider two deterministic consumption plans that yield identical outcomes from the third period onward. Time separability requires that their ranking does not depend on the common continuation.

Axiom I.7 (Stationarity). For all $x \in C$ and $h, g \in \mathbf{H}$, $h \succsim g$ if and only if $(x, h) \succsim (x, g)$.

Stationarity expresses Koopmans's idea that "the passage of time does not have an effect on preferences." It is well known from Koopmans (1972) that, together with time separability and continuity, stationarity implies that the intertemporal utility index is additive. Thus, the ranking of consumption in any given subset of time periods is independent of consumption levels in all other time periods.

In order to talk about ambiguity attitudes we will have to impose some restrictions on the curvature of the utility function that describes consumer's preferences over deterministic alternatives. The next axiom will guarantee the affinity of such utility function, allowing us to identify the translation invariance of our certainty equivalents as an expression of constant absolute ambiguity aversion.

Axiom I.8 (Independence for Deterministic Prospects). For all $d, d', d'' \in \mathbf{D}$ and $\alpha \in (0, 1)$,

$$d \sim d' \implies \alpha d + (1 - \alpha)d'' \sim \alpha d' + (1 - \alpha)d''.$$

Before presenting the next axiom, we will first define the concept of a one-step-ahead consumption plan.

Definition 3. A consumption plan $h \in \mathbf{H}$ is said to be one-step-ahead if $h_t = f$ for some \mathcal{G}_1 -measurable function $f : \Omega \rightarrow C$ and all $t \geq 1$.

We will refer to such plans as one-step-ahead consumption plans. In words, one-step ahead consumption plans resolve all the uncertainty at $t = 1$ and pay a stream of consumption in \mathbf{D} thereafter.

Axiom I.9 (One-step-ahead equivalence). For all $h \in \mathbf{H}$, there exists a one-step-ahead consumption plan $h^+ \in \mathbf{H}$ such that $h_0 = h_0^+$, $h \sim h^+$, and

$$(h^+)^s \sim h^s$$

for all $s \in S$.

This axiom basically requires the existence of a one-step ahead certainty equivalent. We will show how the combination of state-time monotonicity and one-step-ahead equivalence yields *dynamic consistency*. More formally, they imply the following axiom.

Axiom I.10 (Dynamic Consistency). For all $h, g \in \mathbf{H}$ with $h_0 = g_0$,

$$(\forall s \in S, h^s \succsim g^s) \implies h \succsim g,$$

if in addition $h^s \succ g^s$ for some $s \in S$, then we have $h \succ g$.

4 Main results

In the following sections we provide two axiomatizations for recursive preferences and their implications. First, we retrieve a recursive representation for preferences satisfying the notion of *state-time monotonicity* presented and discussed in the previous section. Second, we show a result analogous to Proposition 4 of Bommier et al. (2017), highlighting how in our setting the combination of *state-monotonicity* and *stationarity* imposes a restriction on the decision maker's behavioral attitudes towards ambiguity, specifically the preferences will satisfy constant absolute ambiguity aversion. Third, we show how even though the combination of these two axioms leads to such a restriction it also provides some benefit. In particular, it is shown how under the stronger monotonicity assumption the ex-ante and one-step-ahead certainty equivalents involved in the representation are related by a condition that we call *generalized rectangularity*. This condition provides a functional equation that is shown to have a unique solution by means of Blackwell's sufficient conditions for contractions. Moreover, we show that generalized rectangularity includes the classic rectangularity condition of Epstein and Schneider (2003b) and we discuss its relation to the *no-gain condition* of Maccheroni et al. (2006b).

4.1 Recursive representations and generalized rectangularity

The following result shows that in our setting these axioms characterize recursive preferences of the form presented in Definition 2.

Theorem 1. *Let \succsim be a binary relation on \mathbf{H} . The following are equivalent,*

- (i) \succsim satisfies axioms I.1, I.2, I.4-I.9.
- (ii) \succsim admits a recursive representation.

The main feature of the representation theorem stated above is that it relies on state-time monotonicity (Axiom I.2). In particular, in the proof of Theorem 1, we demonstrate that the combination of state-time monotonicity and one-step-ahead equivalence axioms implies a standard notion of dynamic consistency (recalled in Axiom I.10). This means that our result breaks down dynamic consistency into two more fundamental axioms, one of which is a more intuitive requirement of monotonicity.

In the following example we provide a preference relation that admits a recursive representation, but that does not satisfy state-monotonicity (Axiom I.3).

Example 1. Assume that $S = \{H, T\}$. Consider recursive preferences with the smooth ambiguity certainty equivalent

$$I_{+1} : \xi \mapsto \phi^{-1} \left(\frac{1}{2} \phi(\mathbb{E}_{P^1}[\xi]) + \frac{1}{2} \phi(\mathbb{E}_{P^2}[\xi]) \right),$$

where $P^1(H) = P^1(T) = \frac{1}{2}$ and $P^2(H) = \frac{2}{3} = 1 - P^2(T)$. Assume $\phi(\cdot) = \sqrt{\cdot}$, $u(x) = \mathbb{E}_x[\cdot]$ and $\beta = 0.1$. Let $h = (1, f, \dots, f, \dots)$ and $g = (0, 10 + f, f, \dots, f, \dots)$ where $f : \Omega \rightarrow C$ satisfies $f(H, H, \dots) = 1$ and equals 0 otherwise. Observe that

$$\sum_{t=0}^{\infty} \beta^t h_t(\omega) = \sum_{t=0}^{\infty} \beta^t g_t(\omega),$$

for every $\omega \in \Omega$. Yet we also have

$$\begin{aligned} V(h) &= 1 + 0.1 \left(\frac{1}{2} \sqrt{\frac{1}{0.9} \cdot \frac{1}{2}} + \frac{1}{2} \sqrt{\frac{1}{0.9} \cdot \frac{2}{3}} \right)^2 \\ &\approx 1.0645 \\ &< 1.0648 \\ &\approx 0.1 \left(\frac{1}{2} \sqrt{10 + \frac{1}{0.9} \cdot \frac{1}{2}} + \frac{1}{2} \sqrt{10 + \frac{1}{0.9} \cdot \frac{2}{3}} \right)^2 = V(g). \end{aligned}$$

thus implying a violation of state-monotonicity. Notice however that h is neither better nor worse than g according to state-time monotonicity.

The next result shows that under state-monotonicity, \succsim satisfies constant absolute ambiguity aversion. In particular, preferences can be represented by means of an ex-ante certainty equivalent I_0 that is translation invariant. Moreover, I_0 is linked to I_{+1} through an equation which we refer to as generalized rectangularity as it is a generalization of the rectangularity of MEU preferences from Epstein and Schneider (2003a).

Theorem 2. *Let \succsim be a binary relation on \mathbf{H} . The following are equivalent,*

- (i) \succsim admits the representation (2) and satisfies Axiom I.3.

(ii) \succsim admits a recursive representation with I_{+1} translation invariant and there exists a translation invariant certainty equivalent $I_0 : B(U(\mathbf{D}), \Omega, \mathcal{G}) \rightarrow \mathbb{R}$ such that \succsim is represented by

$$h \mapsto I_0 \left(\sum_{t=0}^{\infty} \beta^t u(h_t) \right).$$

Moreover, I_0 satisfies

$$I_0(\xi) = \beta I_{+1} \left(\frac{1}{\beta} I_0(\xi^1) \right), \quad (3)$$

where $I_0(\xi^1) : s \in S \mapsto I_0(\xi^{s,1})$ for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$.

Outline of the proof. To show the “only if” part, we use state-monotonicity and continuity for the existence of a certainty equivalent I_0 such that the mapping

$$h \mapsto I_0 \left(\sum_{t=0}^{\infty} \beta^t u(h_t) \right) \quad (4)$$

represents \succsim . We show that I_0 must satisfy translation invariance by stationarity. Using translation invariance, (4) and (2), we show that I_0 and I_{+1} are related to each other by (3), which implies that I_{+1} is also translation invariant. The details are in the Appendix. \square

As a consequence, we observe immediately that under state-monotonicity recursive preferences satisfy constant absolute ambiguity aversion.

Corollary 1. *Suppose that \succsim admits the representation (2) and satisfies Axiom I.3. Then \succsim exhibits constant absolute ambiguity aversion.*

Proof. The statement follows immediately by applying Theorem 2 and Proposition 1. \square

A few comments are in order. First, as previously discussed, this result highlights an important trade-off in modeling. If one desires to represent preferences through ex-ante certainty equivalents, it becomes impossible to simultaneously have recursive preferences that abide by decreasing absolute ambiguity aversion. Our suggestion is that to align with this evidence, state-time monotonicity should be adopted as the standard for monotonicity, instead of state monotonicity. This approach accommodates more versatile ambiguity attitudes while retaining a degree of tractability. On

the other hand, we also show allowing for state-monotonicity—or better, for constant absolute ambiguity aversion—yields a useful condition that we call *generalized rectangularity* (3). This condition follows the idea of seeing rectangularity as a generalized form of law of iterated expectations. The primary distinction from a conventional law of iterated expectations is its dependence on the degree of impatience. However, this dependence vanishes when the certainty equivalents are positively homogeneous. As mentioned in the introduction, this functional equation enables us to solve for the ex-ante certainty equivalent I_0 using the one-step-ahead certainty equivalent I_{+1} . The following result demonstrates that, with a fixed I_{+1} , there exists a unique I_0 that satisfies equation (3). Notably, a standard numerical procedure based on the contraction mapping theorem can determine such I_0 . The interpretation is that given a one-step-ahead certainty equivalent, under recursivity and monotonicity one can recover uniquely the ex-ante certainty equivalent.

Theorem 3. *Fix a translation invariant one-step-ahead certainty equivalent $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \rightarrow \mathbb{R}$. There exists a unique translation invariant certainty equivalent $I_0^* : B(U(\mathbf{D}), \Omega, \mathcal{G}) \rightarrow \mathbb{R}$ such that*

$$I_0^*(\xi) = \beta I_{+1} \left(\frac{1}{\beta} I_0^* (\xi^1) \right)$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. Moreover, I_0^* is globally attracting.

Outline of the proof. Using the fact that I_{+1} is translation invariant, this result follows by applying a version of Blackwell’s contraction mapping theorem to a suitably chosen operator. The complete proof is available in the Appendix. \square

In general, however, one should not expect an explicit solution of this functional equation. For example, assume that

$$I_{+1}(\xi) = \int \xi dv_{+1} \text{ for all } \xi \in B(U(\mathbf{D}), S, 2^S),$$

where $v_{+1} : 2^S \rightarrow [0, 1]$ is a capacity that is neither convex nor concave. In this case, there need not be a capacity $v_0 : \mathcal{G} \rightarrow [0, 1]$ such that generalized rectangularity holds, which in this case is equivalent to

$$\int \xi dv_0 = \int \int \xi^1 dv_0 dv_{+1},$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$.⁷ Nevertheless, our finding still enables an analyst interested in retrieving I_0 to do so using numerical techniques. To illustrate, in the Choquet case one can still find a certainty equivalent I_0^* that satisfies

$$I_0^*(\xi) = \int I_0^*(\xi^1) dv_{+1}.$$

4.2 Generalized rectangularity for MEU and variational preferences

In this subsection we show how generalized rectangularity behaves in the case of maxmin and variational preferences. In particular, we show how it is equivalent to the original rectangularity from Epstein and Schneider (2003a), and Epstein and Schneider (2003b) for recursive MEU preferences and its connection with the no-gain condition introduced by Maccheroni et al. (2006b).

Before providing the results we will need some more notation. We recall that $\mathcal{G} = \sigma(\bigcup_{t=1}^{\infty} \mathcal{G}_t)$. For all $A \in \mathcal{G}$ and $s \in S$, let

$$A_s = \{(s_t)_2^\infty : (s, (s_t)_2^\infty) \in A\}.$$

Further, given $P \in \Delta(\Omega)$ and $s \in S$, P_{+1} denotes the marginal over the first coordinate while P_s denotes the marginal over the cylinder set $\{s^\infty \in \Omega : s_1 = s\}$. For our present specification rectangularity takes the following form. Let $\mathcal{L} \subseteq \Delta(S)$ and $\mathcal{P} \subseteq \Delta(\Omega)$. We say that \mathcal{P} is \mathcal{L} -rectangular if $P \in \mathcal{P}$ if and only if there exist $\ell \in \mathcal{L}$ and $\{Q^s \in \mathcal{P} : s \in S\}$ such that

$$P(A) = \sum_{s \in S} \ell(s) Q^s(A_s)$$

for all $A \in \mathcal{G}$. This is the classic notion of rectangularity as introduced by Epstein and Schneider (2003a). We have the following characterization.

Corollary 2 (Rectangularity for MEU⁸). *Suppose that $\mathcal{P} \subseteq \Delta(\Omega)$ and $\mathcal{L} \subseteq \Delta(S)$ are convex and weak* compact sets. If*

$$I_{+1}(\xi) = \min_{P \in \mathcal{P}} \mathbb{E}_P[\xi] \quad \text{for all } \xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$$

⁷We refer to Zimmer (2011) and Dominiak (2013) for cases in which it does or does not hold.

⁸See also Epstein and Schneider (2003a) equations 2.4 and 2.6.

and

$$I_0(\xi) = \min_{\ell \in \mathcal{L}} \mathbb{E}_\ell[\xi] \quad \text{for all } \xi \in B(U(\mathbf{D}), S, \Sigma),$$

then the following are equivalent

1. I_{+1} and I_0 satisfy (3);
2. \mathcal{P} and \mathcal{L} satisfy

$$\min_{P \in \mathcal{P}} P(A) = \min_{\ell \in \mathcal{L}} \left[\sum_{s \in S} \ell(s) \min_{P \in \mathcal{P}} P(A_s) \right] \quad \text{for all } A \in \mathcal{G}; \quad (5)$$

3. \mathcal{P} is \mathcal{L} -rectangular.

Proof. See the Appendix. □

A further major implication is that under state-monotonicity, the only recursive preferences that satisfy uncertainty aversion are variational. We say that $c : \Delta(\Omega) \rightarrow [0, \infty]$ is *grounded* if its infimum value is zero. In addition, we say that c is a *cost function* if it is convex, grounded, and lower semicontinuous.

Corollary 3. *Suppose that \succsim admits the representation (2) and satisfies*

$$h \sim g \Rightarrow \alpha h + (1 - \alpha)g \succsim h, \quad (6)$$

for all $h, g \in \mathbf{H}$ and $\alpha \in (0, 1)$. Then \succsim satisfies state-monotonicity if and only if there exist cost functions $c_{+1} : \Delta(S) \rightarrow [0, \infty]$, $c_0 : \Delta(\Omega) \rightarrow [0, \infty]$ such that for all $h \in \mathbf{H}$,

$$V(h) = u(h_0) + \beta \min_{\ell \in \Delta(S)} \left\{ \mathbb{E}_\ell[V \circ h^1] + c_{+1}(\ell) \right\}$$

and

$$I_0 \left(\sum_{t=0}^{\infty} \beta^t u(h_t) \right) = \min_{P \in \Delta(\Omega)} \left\{ \mathbb{E}_P \left[\sum_{t=0}^{\infty} \beta^t u(h_t) \right] + c_0(P) \right\}.$$

Moreover, a sufficient condition for (3) is given by

$$c_0(P) = \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1}), \quad (7)$$

for all $P \in \Delta(\Omega)$.

Outline of the proof. Because of (6) and state-monotonicity, I_{+1} is translation invariant and quasi-concave. Likewise, there exists I_0 that is translation invariant and quasi-concave. Hence by standard results the certainty equivalents I_{+1} and I_0 have the desired variational representation. It then follows that generalized rectangularity is implied by (7). See the Appendix for the full proof. \square

Observe that (7) is reminiscent of the no-gain condition in Maccheroni et al. (2006b). In this setting, the no-gain condition is sufficient only because we do not have unbounded utility. We discuss more technical points related to the necessity of (7) in the Appendix. In particular, under weak technical requirements the inequality

$$c_0(P) \leq \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1}), \quad (8)$$

is implied by generalized rectangularity. This result further illustrates how our generalized rectangularity subsumes the major characterizations of recursive beliefs.

The next example, based on entropic cost functions, provides a notable case of cost function that satisfy (7). Given $Q \in \Delta^\sigma(Y)$, the relative entropy $R(\cdot \| Q)$ is a mapping from $\Delta(Y)$ into $[0, \infty]$ defined by

$$R(P \| Q) = \begin{cases} \int_Y \left(\log \frac{dP}{dQ} \right) dP, & \text{if } P \in \Delta^\sigma(Y) \text{ and } P \ll Q \\ \infty & \text{otherwise.} \end{cases}$$

Example 2. Given $Q \in \Delta^\sigma(\Omega)$, let

$$c_0(P) = \frac{1}{\theta} R(P \| Q) \quad \text{for every } P \in \Delta^\sigma(\Omega),$$

and

$$c_{+1}(\ell) = \frac{1}{\beta\theta} R(\ell \| Q_{+1}) \quad \text{for every } \ell \in \Delta^\sigma(S).$$

This entropic cost formulation corresponds to recursive multiplier preferences (see Strzalecki (2011) for the static formulation). Then (7) reduces to

$$R(P \| Q) = \sum_{s \in S} P_{+1}(s) R(P_s \| Q_s) + R(P_{+1} \| Q_{+1}),$$

which can be shown to be always satisfied for every $P \in \Delta(\Omega)$ with the same reasoning used in Maccheroni et al. (2006b) (see their Theorem 3).

4.3 Sequential choice

As mentioned, we follow Epstein’s terminology (see Epstein (1992)) and therefore distinguish sequential choice problems from intertemporal consumption choice problems. The former models situations taking place over short intervals of time during which consumption/savings plans can be taken to be fixed. The source of utility is terminal wealth rather than a consumption sequence. In the latter, consumption decisions take place over a long period of time, and the source of utility is given by a consumption stream.

In sequential choice problems, the decision maker (DM) has to choose a bet ex-ante and at an interim information stage, where the DM can bet based on partial resolution of uncertainty (see for example Section 1.1 in Hanany and Klibanoff (2009) for an example). Subsequently, a state realizes and payment occurs depending on the DM’s choice. Figure 4 offers a graphical representation of a sequential choice problem.

This setting builds upon the one on static choice problems presented above. In particular, there is a finite partition Π of Ω , and $\mathcal{G} := \sigma(\Pi)$ is the σ -algebra generated by the partition Π . Given $\omega \in \Omega$, we denote by $\Pi(\omega)$ the unique element of Π such that $\omega \in \Pi(\omega)$. The set of acts is denoted by \mathbf{F} and the DM’s preferences are expressed as unconditional and conditional total preorders over \mathbf{F} . Such preferences are denoted by $\langle \succsim, (\succsim_E)_{E \in \Pi} \rangle$. The binary relation \succsim represents the ex-ante preferences of the DM while \succsim_E models the DM’s preferences conditional on the event $E \in \Pi$. We

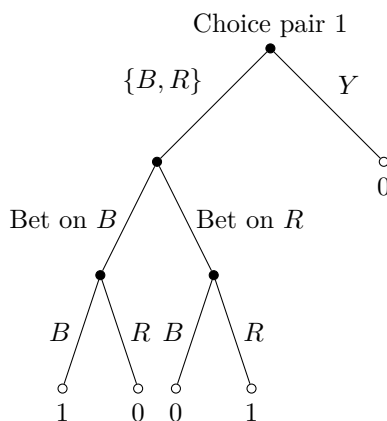


Figure 4: An Ellsberg type sequential choice problem

introduce a set of basic rationality axioms for \succsim .

Axiom S.1 (Monotonicity). For all $f, g \in \mathbf{F}$,

$$[\forall \omega \in \Omega, f(\omega) \succsim g(\omega)] \implies f \succsim g,$$

and if $f(\omega) \succ g(\omega)$ for some $\omega \in \Omega$, then $f \succ g$.

Axiom S.2 (Mixture continuity). For all $f, g, h \in \mathbf{F}$ the sets

$$\{\lambda \in [0, 1] : \lambda f + (1 - \lambda)g \succsim h\} \text{ and } \{\lambda \in [0, 1] : h \succsim \lambda f + (1 - \lambda)g\},$$

are closed.

Axiom S.3 (Risk independence). For all $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$x \succsim y \iff \alpha x + (1 - \alpha)z \succsim \alpha y + (1 - \alpha)z,$$

The next two axioms refer to preferences $\langle \succsim, (\succsim_E)_{E \in \Pi} \rangle$ and link them to recursivity.

Axiom S.4 (Dynamic consistency (DC)). For all $f, g \in \mathbf{F}$,

$$[\forall E \in \Pi, f \succsim_E g] \implies f \succsim g.$$

For all $f, g \in \mathbf{F}$ and all $E \in \Pi$, denote by fEg the act equal to f when restricted to E , and equal to g otherwise.

Axiom S.5 (Consequentialism (C)). For all $E \in \Pi$ and all $f, g \in \mathbf{F}$,

$$f \sim_E fEg.$$

We call *recursive* those preferences that satisfy axioms S.1-S.5. The term recursivity is justified from the fact that such axioms characterize the following representation.

Theorem 4. *Let \succsim be a binary relation on \mathbf{F} . The following are equivalent,*

(i) \succsim satisfies axioms S.1-S.5.

(ii) *There exist an affine function $u : X \rightarrow \mathbb{R}$ and certainty equivalents $I : B_0(u(X), \Omega, \mathcal{G}) \rightarrow \mathbb{R}$, and $\bar{I}(\omega, \cdot) : B_0(u(X), \Omega, \mathcal{G}) \rightarrow \mathbb{R}$ for all $\omega \in \Omega$ such that*

$$(a) \ I(u(f)) = I(\bar{I}(\cdot, u(f))) \text{ for all } f \in \mathbf{F},$$

$$(b) \ \bar{I}(\omega, u(f)1_{\Pi(\omega)} + u(g)1_{\Pi(\omega)^c}) = \bar{I}(\omega, u(f)) \text{ for all } \omega \in \Omega \text{ and all } f, g \in \mathbf{F},$$

(c) $f \mapsto I(u(f))$ represents \succsim ,

(d) for all $E \in \Pi$ and $\omega \in E$ the mapping $f \mapsto I(\omega, u(f))$ represents \succsim_E .

Proof. We omit the proof of this classic result. A proof can be found for example in Cerreia-Vioglio et al. (2022), Proposition 8. \square

In this setting, however, constant absolute ambiguity aversion is not implied by recursivity, as shown in the next example.

Example 3. Consider \succsim represented by

$$I(u(f)) = \phi^{-1} \left(\int \phi(u(f)) d\mu \right),$$

for all $f \in \mathbf{F}$ and \succsim_E is represented by $I_E(u(f)) = \phi^{-1} (\int \phi(u(f)) d\mu_E)$. In this case DC and C are satisfied for all invertible ϕ even when ϕ is not an exponential, such as in the CRRA case. Indeed,

$$\begin{aligned} I(I_E(u(f))) &= I \left(\phi^{-1} \left(\int \phi(u(f)) d\mu_E \right) \right) \\ &= \phi^{-1} \left(\int \left(\int \phi(u(f)) d\mu_E \right) d\mu \right) \\ &= \phi^{-1} \left(\int \phi(u(f)) d\mu \right). \end{aligned}$$

by the law of iterated expectations. Therefore, the function ϕ can have any shape, thus allowing for arbitrary attitudes toward uncertainty. For example, if $\phi(x) = x^\rho$ for $0 < \rho < 1$, then \succsim will not satisfy CAAA over \mathbf{F} .

However, as shown by Savochkin et al. (2022) CAAA is implied by recursivity when \succsim belongs to the smooth ambiguity class, i.e., when preferences admit the representation

$$V : f \mapsto \phi^{-1} \left(\int_{\Delta(\Omega)} \phi \left(\int_{\Omega} u(f(\omega)) d\mu \right) d\pi(\mu) \right),$$

for some concave and differentiable function ϕ . Their results imply that ϕ has to be either linear or such that $\phi(x) = -e^{-\theta x}$ for some $\theta > 0$. Our work emphasizes that the key behavioral restriction entailing constant absolute ambiguity aversion is the following

$$[\forall E \in \Pi, f \sim_E g] \Rightarrow [\forall E \in \Pi, fEh \sim gEh],$$

which is a joint implication of Axioms S.4 and S.5. In their proof, step 3a (p. 23 of their working paper), such a property is used to derive a functional equation that implies that ϕ is an exponential, therefore satisfying CAAA.

5 Discussion

5.1 Related literature

Bommier et al. (2017) is the theoretical work closest to the present paper. Similar to our approach, they examine recursive preferences that satisfy state-monotonicity, which they refer to as just monotonicity. However, there are notable differences between their study and ours. They do not investigate the implications of state-monotonicity for general ambiguity attitudes as we do. Additionally, our work diverges from theirs in terms of methodology; we do not employ their techniques based on the theory of functional equations to derive translation invariance of the one-step ahead certainty equivalent I_{+1} . Furthermore, we assume time-separable preferences, whereas they do not. In a related work, Li et al. (2023) investigate various forms of monotonicity when preferences are defined over matrices. Their notion of outcome monotonicity is analogous to our concept of state-time monotonicity in the current setting. They demonstrate that the various forms of monotonicity they consider hold jointly if and only if preferences can be represented by discounted expected utility.

So far, we have discussed how ambiguity attitudes evolve when a decision maker becomes better off in terms of utility. Alternatively, one may want to predict changes in ambiguity attitude when the decision maker becomes better off in terms of wealth. This approach requires to account for risk attitudes, as shown by Cerreia-Vioglio et al. (2019). We leave this to future research. For the moment, we observe that our results still apply using Cerreia-Vioglio et al.’s methodology under the assumption of risk neutrality.

Savochkin et al. (2022) consider a setting of sequential choice and characterize recursive smooth ambiguity preferences. While the major source of appeal of the smooth ambiguity model is that it need not satisfy constant ambiguity aversion (either absolute or relative), they show that CAAA is necessary under the assumption of recursivity. Further, they derive a condition for the decision maker’s beliefs that ensures recursivity.

Baillon and Placido (2019) and Berger and Bosetti (2020) provide experimental evidence on non-constant ambiguity aversion. In particular, Baillon and Placido’s results call for the use of ambiguity models that can accommodate decreasing aversion toward ambiguity. In the context of intertemporal consumption choice problems, we

proposed a novel notion of monotonicity to address this point.

5.2 Concluding remarks

Our paper loosely suggest a result of the following type: one cannot have at the same time

1. preferences that are recursive and hence tractable;
2. a representation of preferences by means of an ex-ante certainty equivalent; and
3. (strictly) decreasing absolute ambiguity aversion, as consistent with the experimental literature.

Under the assumption of points (1) and (2) above, we have provided a full characterization of recursive preferences, establishing in particular a generalized notion of rectangularity of beliefs. At the same time for intertemporal consumption choice problems—the main focus of the applied literature—we suggest a novel notion of monotonicity, namely state-time monotonicity. State-time monotonicity is compatible with discarding point (2) from the list above and allows for the examination of tractable preferences in a dynamic setting while accommodating realistic uncertainty attitudes. The only minor trade-off from a decision-theoretic standpoint is the absence of an ex-ante certainty equivalent to represent preferences.

Appendix

Proofs of the main results

We provide the proof in several steps starting with the derivation of discounted utility representation $d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t)$ for \succsim over the set of deterministic processes \mathbf{D} . We detail the proof to make our paper as self-contained and explicit as possible. In particular, we follow the approach of Bastianello and Faro (2022). Consider the following axioms:

(P1) (Continuity) For all compact sets $K \subseteq C$ and all $d \in \mathbf{D}$, the sets $\{d' \in K^\infty \mid d \succsim d'\}$ and $\{d' \in K^\infty \mid d' \succsim d\}$ are closed in the product topology over K^∞ .

(P2) (Sensitivity) There exist $x, y \in C$, $d \in \mathbf{D}$ such that $(x, d) \succ (y, d)$.

(P3) (Stationarity) For all $x \in C$ and $d, d' \in \mathbf{D}$, $d \succsim d'$ if and only if $(x, d) \succsim (x, d')$.

(P4) (Time separability) For all $x, y, x', y' \in C$ and $d, d' \in \mathbf{D}$, $(x, y, d) \sim (x', y', d)$ if and only if $(x, y, d') \sim (x', y', d')$.

(P5) (Monotonicity) Let $d, d' \in \mathbf{D}$. If $d_t \succsim d'_t$ for all $t \geq 0$, then $d \succsim d'$; if moreover $d_t \succ d'_t$ for some $t \geq 0$ then $d \succ d'$.

Proposition 2 (Bastianello and Faro (2022)). *A preference relation \succsim over \mathbf{D} satisfies P.1-P.5 if and only if there exists a continuous function $u : C \rightarrow \mathbb{R}$ and a discount factor $\beta \in (0, 1)$ such that \succsim is represented by*

$$U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t).$$

Proof. See Proposition 5 in Bastianello and Faro (2022). □

Lemma 1. *Axioms I.1, I.2, I.4, I.5, and I.7 imply (P2)*

Proof. The proof is analogous to the Lemma 5 in Kochov (2015); we report it here for the sake of completeness. Suppose that $(x, d) \sim (y, d)$ for all $x, y \in C$ and all $d \in \mathbf{D}$, then by stationarity $(z, x, d) \sim (z', x, d) \sim (z', y, d)$ for all $z, z', x, y \in C$ and $d \in \mathbf{D}$. Repeating this argument we have that $d \sim d'$ for all $d, d' \in \mathbf{D}$ differ in at most finitely many points. Let $d = (x_0, x_1, \dots)$ and $d' = (y_0, y_1, \dots)$ in \mathbf{D} and define $d^t = (x_0, \dots, x_{t-1}, y_t, y_{t+1})$. The previous argument shows that $d^t \sim d$ for all $t \geq 0$ and $(d^t)_{t=1}^{\infty}$ converges to d' . By continuity and completeness of \succsim we have $d \sim d'$ because $d, d' \in \mathbf{D}$ were arbitrarily chosen and we have a contradiction of Axiom I.5 since $\bar{c} > \underline{c}$. □

Lemma 2 (Bastianello and Faro (2022)). *Axioms I.1, I.4, I.6, and I.7 imply (P5).*

Proof. See Lemma A.2 in Bastianello and Faro (2022). □

Lemma 3. *A preference relation \succsim over \mathbf{D} axioms I.1, I.2, and I.4-I.8 if and only if there exists a continuous and affine function $u : C \rightarrow \mathbb{R}$ and a discount factor $\beta \in (0, 1)$ such that \succsim is represented by*

$$U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t).$$

Proof. By Lemmas 1 and 2 we have that axioms I.1, I.2, and I.4-I.8 imply **(P2)** and **(P5)**. It is immediately observable that all the others **(P1)**, **(P3)**, **(P4)** are directly implied by axioms I.1, I.2, and I.4-I.8. Therefore, by Proposition 2 we have that there exists a continuous function $u : C \rightarrow \mathbb{R}$ and a discount factor $\beta \in (0, 1)$ such that $U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t)$ represents \succsim . Now notice that when restricted to C the preference relation \succsim satisfies all the hypotheses of Theorem 8 in Herstein and Milnor (1953); therefore \succsim admits an affine utility representation v . Since v must be cardinally unique it follows that u must be a positive affine transformation of v and as such it must be affine. \square

Before the proof of the next result we need the following definition.

Definition 4 (*t*-step-ahead plan). *A consumption plan $h \in \mathbf{H}$ is a t-steps-ahead plan if h_τ is measurable with respect to \mathcal{G}_t for every $\tau \geq 1$.*

Let \mathbf{H}_t denote the set of *t*-step-ahead plans.

Proof of Theorem 1. [(ii) \Rightarrow (i)]. Checking that the axioms are necessary for the representation is routine, except for the one-step-ahead equivalence and state-time monotonicity, whose necessity we now show. To show that, take any $h \in \mathbf{H}$ and let $(c^s)_{s \in S} \in C^S$ be such that $U(c^s, c^s, \dots) = V(h^s)$ for all $s \in S$. Define $h_0^{+1} = h_0$ and $h_t^{+1} = f$ for all $t \geq 0$ where $f : S \rightarrow C$ with $f(s) = c^s$ for all $s \in S$. Observe that by construction we have $h_0 = h_0^{+1}$ and $(h^{+1})^s \sim h^s$ for all $s \in S$. Since \succsim admits a recursive representation, it must satisfy dynamic consistency, and hence $(h^{+1})^s \sim h^s$ for all $s \in S$ and $h_0 = h_0^{+1}$ imply $h \sim h^{+1}$.

Turning to state-time monotonicity, we use the following notion to prove that it is satisfied. We divide the proof in three steps. Take h, g such that $h \geq_{STM} g$, that is

$$\forall \omega \in \Omega, t \geq 0, h_t(\omega) \succsim g_t(\omega).$$

Step 1. The statement holds $h \succsim g$ for every $h, g \in \mathbf{H}_1$. Indeed, $V(h^{s,1}) \geq V(g^{s,1})$ for every $s \in S$, so that the statement follows by monotonicity of I_{+1} .

Step 2. If $h, g \in \mathbf{H}_t$, then there exist $\hat{h}, \hat{g} \in \mathbf{H}_{t-1}$ such that $h \sim \hat{h}, g \sim \hat{g}$ and $\hat{h} \geq_{STM} \hat{g}$. Indeed, for every $s^{t-1} = (s_1, \dots, s_{t-1}) \in S^{t-1}$ with $t \geq 2$, using the recursive representation one can find $x(s_1, \dots, s_{t-1}), x'(s_1, \dots, s_{t-1}) \in C$ satisfying

$$(h)^{s^{t-1}, t-1} \sim (x(s_1, \dots, s_{t-1}), x(s_1, \dots, s_{t-1}), \dots),$$

and

$$(g)^{s^{t-1}, t-1} \sim (x'(s_1, \dots, s_{t-1}), x'(s_1, \dots, s_{t-1}), \dots).^9$$

Now define \hat{h} and \hat{g} as follows

$$\hat{h}_\tau(s_1, \dots, s_\tau) = \begin{cases} h_\tau(s_1, \dots, s_\tau) & \tau \leq t-1, \\ x(s_1, \dots, s_{t-1}) & \tau > t-1, \end{cases}$$

and

$$\hat{g}_\tau(s_1, \dots, s_\tau) = \begin{cases} g_\tau(s_1, \dots, s_\tau) & \tau \leq t-1, \\ x'(s_1, \dots, s_{t-1}) & \tau > t-1. \end{cases}$$

It follows that \hat{h} and \hat{g} are $t-1$ -step-ahead plans such that $\hat{h} \geq_{SMT} \hat{g}$, $\hat{h} \sim h$ and $\hat{g} \sim g$.¹⁰

Step 3. If $h, g \in \mathbf{H}_t$, then $h \succsim g$. By Step 2, there exist $\hat{h}, \hat{g} \in \mathbf{H}_{t-1}$ such that $h \sim \hat{h}$, $g \sim \hat{g}$ and $\hat{h} \geq_{STM} \hat{g}$. Re-applying the same step to \hat{h}, \hat{g} and forward we have that there exist $\bar{h}, \bar{g} \in \mathbf{H}_1$ such that $h \sim \bar{h}$, $g \sim \bar{g}$ and $\bar{h} \geq_{STM} \bar{g}$. Thus, by Step 1, we have that $h \succsim g$.

Step 4. Observe that for every $h, g \in \mathbf{H}$ and $h \geq_{STM} g$, given $x \in C$ by defining h^t, g^t as follows

$$h_\tau = \begin{cases} h_\tau & \tau \leq t, \\ x & \tau > t, \end{cases}$$

$$g_\tau = \begin{cases} g_\tau & \tau \leq t, \\ x & \tau > t, \end{cases}$$

we have that $(h^t(\omega))_{t=0}^\infty$ and $(g^t(\omega))_{t=0}^\infty$ converge to $h(\omega)$ and $g(\omega)$ for every $\omega \in \Omega$, respectively. Since all h^t, g^t belong to \mathbf{H}_t and $h^t \geq_{STM} g^t$, by Step 3, we have that $h^t \succsim g^t$ for every $t \geq 1$. Therefore by continuity of \succsim we obtain $h \succsim g$ as desired.

[(i) \Rightarrow (ii)]. First observe that by state-time monotonicity and the one-step-ahead equivalence axioms we have that for every $h, g \in \mathbf{H}$ such that $h_0 = g_0$

$$(\forall s \in S, h^s \succsim g^s) \implies h \succsim g. \quad (9)$$

⁹Here $(h)^{s^{t-1}, t-1}$ denotes the plan $(h_{t-1}(s^{t-1}), h_t(s^{t-1}, \cdot), \dots)$.

¹⁰The proof of the previous equivalences relies on the re-iteration of the representation till time t and the application of stationarity. These details are available upon request.

Indeed, suppose that $h, g \in \mathbf{H}$ are such that $h_0 = g_0$ and $h^s \succsim g^s$ for all $s \in S$. By one-step ahead equivalence, there exist one-step ahead acts h^{+1} and g^{+1} such that $h \sim h^{+1}$, $g \sim g^{+1}$, $(h^{+1})^s \sim h^s$, and $(g^{+1})^s \sim g^s$ for all $s \in S$. Fix $s \in S$, $t, t' \geq 0$, and $\omega, \omega' \in \{s\} \times S^\infty$. Notice that since h^{+1} is a one-step ahead act, we must have

$$h_t^{+1}(\omega) = h_{t'}^{+1}(\omega')$$

and hence, given the arbitrariness of t, t' and ω, ω' we have

$$(h^{+1})^s = (h_0, h_t^{+1}(\omega), h_t^{+1}(\omega), \dots)$$

for all $t \geq 0$ and $\omega \in \{s\} \times S^\infty$. The same reasoning applies to g^{+1} . Moreover, by transitivity we have

$$(h^{+1})^s \sim h^s \succsim g^s \sim (g^{+1})^s$$

for all $s \in S$. Given the arbitrariness of $s \in S$ in the previous observations, we have that

$$(h_0, h_t^{+1}(\omega), h_t^{+1}(\omega), \dots) \succsim (h_0, g_t^{+1}(\omega), g_t^{+1}(\omega), \dots)$$

for all $\omega \in \Omega$ and $t \geq 0$. By stationarity, it follows that

$$(h_t^{+1}(\omega), h_t^{+1}(\omega), \dots) \succsim (g_t^{+1}(\omega), g_t^{+1}(\omega), \dots)$$

for all $\omega \in \Omega$ and $t \geq 0$. Then, by state-time monotonicity and transitivity we have that

$$h \sim h^{+1} \succsim g^{+1} \sim g.$$

Further, by Lemma 3 there exists an affine function $u : C \rightarrow \mathbb{R}$ such that

$$U : d \mapsto \sum_{t=0}^{\infty} \beta^t u(d_t),$$

represents \succsim on \mathbf{D} with $\beta \in (0, 1)$. Fix $h \in \mathbf{H}$. Then, by state-time monotonicity and monotonicity on consumption levels we have that the sets

$$\{c \in [\underline{c}, \bar{c}] : (h_0, c, \dots, c, \dots) \succsim h\},$$

and

$$\{c \in [\underline{c}, \bar{c}] : h \succsim (h_0, c, \dots, c, \dots)\},$$

are non-empty. Furthermore, by the continuity they are both closed. Since \succsim is a weak order, it holds

$$\{c \in [\underline{c}, \bar{c}] : (h_0, c, \dots, c, \dots) \succsim h\} \cup \{c \in [\underline{c}, \bar{c}] : h \succsim (h_0, c, \dots, c, \dots)\} = [\underline{c}, \bar{c}].$$

Therefore, since $[\underline{c}, \bar{c}]$ is connected, we must have that

$$\{c \in [\underline{c}, \bar{c}] : (h_0, c, \dots, c, \dots) \succsim h\} \cap \{c \in [\underline{c}, \bar{c}] : h \succsim (h_0, c, \dots, c, \dots)\} \neq \emptyset.$$

which implies that there exists $c_h \in [\underline{c}, \bar{c}]$ such that $(h_0, c_h, \dots, c_h, \dots) \sim h$. Thus we can define the map

$$V(h) = u(h_0) + \beta U((c_h, \dots, c_h, \dots))$$

for all $h \in \mathbf{H}$ and V represents \succsim . Now define $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \rightarrow \mathbb{R}$ as $I_{+1}(V(h^1)) = U((c_h, \dots, c_h, \dots))$. It is straightforward to verify that I_{+1} is well-defined and normalized.¹¹ Now we prove that I_{+1} is also monotone. Suppose that $\xi \geq \xi'$ for some $\xi, \hat{\xi} \in B(U(\mathbf{D}), S, \Sigma)$. Given that I_{+1} is independent of the first period 0, we can assume without loss of generality that there exist $h, \hat{h} \in \mathbf{H}$ such that $h_0 = \hat{h}_0$ and $\xi = V(h^1)$, $\hat{\xi} = V(\hat{h}^1)$. Since V represents \succsim , we have that $h^{s,1} \succsim \hat{h}^{s,1}$ for all $s \in S$. But then, by (9) we have that $(h_0, h^1) \succsim (h_0, \hat{h}^1)$, that is $h \succsim \hat{h}$. Therefore we obtain $V(h) \geq V(\hat{h})$ which implies that $U(c_h) \geq U(c_{\hat{h}})$, thus delivering the desired monotonicity of I_{+1} . The continuity of I_{+1} follows from the continuity of \succsim . Hence, we obtain the desired recursive representation

$$V : h \mapsto u(h_0) + \beta I_{+1}(V(h^1)).$$

□

Proof of Theorem 2. [(i) \Rightarrow (ii)]. By the representation (2) we have that for all $d \in \mathbf{D}$

$$U(d) = \sum_{t=0}^{\infty} \beta^t u(d_t).$$

Normalize u so that $0 \in U(\mathbf{D})$. Now notice that, by continuity and state-monotonicity, for all $h \in \mathbf{H}$ there exists $d^h \in \mathbf{D}$ such that $h \sim d^h$. We construct I_0 as follows (see also the discussion in Bommier et al. (2017), pp. 1445-1449).

¹¹Notice in particular, that the following holds true

$$B(U(\mathbf{D}), S, \Sigma) = \{V(h^1) : h \in H\}.$$

Step 1. For every $t \geq 1$ let

$$B_t := \left\{ \xi \in B(U(\mathbf{D}), \Omega, \mathcal{G}) : \xi = U(x^{t-1}, h), x^{t-1} \in C^{t-1}, h \in \mathbf{H}_1 \right\},$$

and observe that by state monotonicity and continuity we can define certainty equivalents $I_t : B_t \rightarrow \mathbb{R}$ as $I_t(\xi) = U(d^{(x^{t-1}, h)})$ where $d^{(x^{t-1}, h)} \in \mathbf{D}$ satisfies $d^{(x^{t-1}, h)} \sim (x^{t-1}, h)$.¹²

Step 2. Now for every $h \in \mathbf{H}_t$ with $\xi = \sum_{\tau=0}^{\infty} \beta^\tau u(h_\tau)$, set $I_0(\xi) = I_1(\xi)$ when $t = 1$ and when $t \geq 2$ define $I_0(\xi)$ as follows: for every (s_1, \dots, s_{t-1}) let $V_{t-1}(s_1, \dots, s_{t-1})$ be defined by

$$V_{t-1}(s_1, \dots, s_{t-1}) = I_t(\xi(s_1, \dots, s_{t-1}, \cdot)),$$

and recursively for $1 < \tau < t$

$$V_{\tau-1}(s_1, \dots, s_{\tau-1}) = I_\tau(V_\tau(s_1, \dots, s_{\tau-1}, \cdot)),$$

and finally set $I_0(\xi) = I_1(V_1(\cdot))$.

Step 3. For every $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ such that $\xi = \sum_{\tau=0}^{\infty} \beta^\tau u(h_\tau)$, consider $h^t = (h_0, \dots, h_t, x, x, \dots)$ where $u(x) = 0$. Observe that $h \in \mathbf{H}_t$. We further have that $\xi^t = \sum_{\tau=0}^t \beta^\tau u(h_\tau)$ converges pointwise to ξ as $t \rightarrow +\infty$. We define $I_0(\xi) = \lim_{t \rightarrow +\infty} I_0(\xi^t)$.

Now observe that for every $t \geq 1$ and $\xi \in B^{t+1}$ such that $\xi = U(x^t, h)$ with $x^t \in C^t$ satisfying $\sum_{\tau=0}^{t-1} \beta^\tau u(x_\tau) = 0$ stationarity implies that $d^{(x^t, h)} \sim (x^t, d^h)$ where $h \sim d^h$. We therefore obtain for every $t \geq 0$

$$I_{t+1}(\xi) = U(x^t, d^h) = \beta^t I_1 \left(\frac{\xi}{\beta^t} \right). \quad (10)$$

Now for every $\xi \in B^{t+1}$ and $k \in U(\mathbf{D})$ such that $\xi + k \in B^{t+1}$, there must be $h \in \mathbf{H}_1$ and $x^t \in C^t$ such that $k + \xi = \sum_{\tau=0}^{t-1} \beta^\tau u(x_\tau) + \sum_{\tau \geq t} \beta^\tau u(h_\tau)$. By stationarity $(x^t, h) \sim (x^t, d^h)$ where $d^h \in \mathbf{D}$ is such that $d^h \sim h$, so that we obtain

$$I_{t+1}(k + \xi) = \sum_{\tau=0}^{t-1} \beta^\tau u(x h_\tau) + \beta^t I_1 \left(\frac{\xi}{\beta^t} \right) = k + I_{t+1}(\xi).$$

In words, this means that each I_{t+1} is translation invariant for every $t \geq 1$. By (10), it follows that I_1 is translation invariant as well. Therefore, by point (2) above,

¹²Observe that by x^t we mean $x^t = (x_0, \dots, x_{t-1}) \in C^t$ under the convention that $(x^0, h) = h$.

we obtain that I_0 satisfies $I_0(\xi + k) = I_0(\xi) + k$ whenever $\xi + k = \sum_{\tau \geq 0} \beta^\tau u(h_\tau)$ for some $h \in \mathbf{H}_t$. Hence, taking ξ^t that converges pointwise to ξ constructed as in point (3) above, we obtain $I_0(\xi + k) = \lim_{t \rightarrow +\infty} I_0(\xi^t + k) = \lim_{t \rightarrow +\infty} I_0(\xi^t) + k = I_0(\xi) + k$, as desired.

Now we prove that I_0 and I_{+1} must satisfy condition (3). To this end fix $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ with $\xi = \sum_{t=1}^{\infty} \beta^t u(h_t)$ for some $h \in \mathbf{H}$.¹³ Since \succsim admits a recursive representation we have that

$$I_0 \left(\sum_{t=0}^{\infty} \beta^t u(h_t) \right) = u(h_0) + \beta I_{+1} (V \circ h^1).$$

Now using the translation invariance of I_0 , we have that

$$I_0 \left(\sum_{t=1}^{\infty} \beta^t u(h_t) \right) = \beta I_{+1} (V \circ h^1). \quad (11)$$

Now observe that (10) implies that

$$V \circ h^1 = \frac{1}{\beta} I_0 (\xi^1),$$

combining this with (11), we obtain

$$I_0(\xi) = \beta I_{+1} \left(\frac{1}{\beta} I_0 (\xi^1) \right). \quad (12)$$

Finally, translation invariance of I_{+1} is implied by generalized rectangularity (12). Indeed, suppose by contradiction that $I_{+1}(\xi + k\mathbf{1}_\Omega) \neq I_{+1}(\xi) + k$ for some $\xi \in B(U(\mathbf{D}), S, \Sigma)$ and $k \in U(\mathbf{D})$ with $\xi + k\mathbf{1}_S \in B(U(\mathbf{D}), S, \Sigma)$. Then it is routine to find $\varphi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$, $w \in U(\mathbf{D})$ such that $\xi = \frac{1}{\beta} I_0(\varphi^1)$ and $k = \frac{w}{\beta}$.¹⁴ Clearly by translation invariance of I_0 , we must have that $\xi + k = I_0 \left(\frac{1}{\beta} (\varphi^1 + w\mathbf{1}_\Omega) \right)$. Thanks

¹³Notice that this is without loss of generality. To deal with a general element $\xi = \sum_{t=0}^{\infty} \beta^t u(h_t)$ we could work directly with the unique monotone, normalized, and translation invariant $\hat{I}_0 : B(U(\mathbf{D}), \Omega, \mathcal{G}) + \mathbb{R} \rightarrow \mathbb{R}$ extension of I_0 . In particular, the subsequent proof would be changed by first proving generalized rectangularity for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ with $\xi = \sum_{t=1}^{\infty} \beta^t u(h_t)$ for some $h \in \mathbf{H}$, then passing to prove the (restricted) translation invariance of I_{+1} with respect to all such ξ . Once done that, one can prove generalized rectangularity for the unique extension \hat{I}_0 using the (restricted) translation invariance I_{+1} . In conclusion, one use the full generalized rectangularity to prove the complete translation invariance of I_{+1} . We avoid these formalities.

¹⁴For instance define $\varphi^1 = \beta\xi$ and use the normalization of I_0

to these observations, applying generalized rectangularity we have

$$\begin{aligned}
I_0(\varphi + w\mathbf{1}_\Omega) &= \beta I_{+1} \left(\frac{1}{\beta} I_0 \left((\varphi^1 + w\mathbf{1}_\Omega) \right) \right) \\
&= \beta I_{+1} (\xi + k\mathbf{1}_S) \\
&\neq \beta I_{+1} (\xi) + \beta k \\
&= \beta I_{+1} \left(I_0 \left(\frac{1}{\beta} \varphi^1 \right) \right) + \beta k \\
&= I_0(\varphi) + w
\end{aligned}$$

contradicting the translation invariance of I_0 .

[(ii) \Rightarrow (i)] Since I_{+1} is translation invariant this part of the proof follows from Proposition 4 (case 1) in Bommier et al. (2017). □

Proof of Theorem 3. We start defining the following set

$$B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G}) := \{I : B(U(\mathbf{D}), \Omega, \mathcal{G}) \rightarrow \mathbb{R} \mid I \text{ is a translation invariant certainty equivalent}\}.$$

Each $I \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ is a translation invariant certainty equivalent and as such it is 1-Lipschitz. More importantly, notice that

$$U(\mathbf{D}) \subseteq \left[\frac{u(\underline{c})}{1-\beta}, \frac{u(\bar{c})}{1-\beta} \right].$$

Since each $I \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ is normalized and monotone, we have that

$$\|I\|_\infty \leq \frac{u(\bar{c})}{1-\beta}.$$

Thus, we can endow $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ with the sup-metric d_∞ , defined as

$$d_\infty(I, I') = \|I - I'\|_\infty$$

for all $I, I' \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$. Now we verify that $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ is a complete metric space with respect to d_∞ . To this end, fix a Cauchy sequence $(I_n)_{n \in \mathbb{N}}$ in $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ and define $I \in \mathbb{R}^{B(U(\mathbf{D}), \Omega, \mathcal{G})}$ as $I(\xi) = \lim_{n \rightarrow \infty} I_n(\xi)$, I is well-defined since for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ we have that $(I_n(\xi))_{n \in \mathbb{N}}$ is a Cauchy sequence in the set of real numbers. For all $k \in U(\mathbf{D})$ we have that $I_n(k) = k$ and hence $I(k) = \lim_{n \rightarrow \infty} k = k$, thus I is normalized. Monotonicity and translation invariance

follow analogously. In particular, if $\xi \geq \xi'$, then $I_n(\xi) \geq I_n(\xi')$ for all $n \in \mathbb{N}$ and thus $I(\xi) \geq I(\xi')$. For translation invariance, fix $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ and $k \in U(\mathbf{D})$ with $\xi + k \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. Then, it follows that

$$I(\xi + k) = \lim_{n \in \mathbb{N}} I_n(\xi) + k = I(\xi) + k.$$

Thus, $I \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$. To conclude we need to show that $I_n \xrightarrow{d_\infty} I$. Let $\varepsilon > 0$ and $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. Then there exists N_ε , such that $d_\infty(I_n, I_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$. This yields

$$|I(\xi) - I_n(\xi)| = \lim_{m \rightarrow \infty} |I_m(\xi) - I_n(\xi)| < \varepsilon$$

for all $n \geq N_\varepsilon$. Since ξ was chosen arbitrarily $d_\infty(I, I_n) < \varepsilon$. Thus, we have that $(B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G}), d)$ is a complete metric space. Now we define the following map,

$$\begin{aligned} T : \mathbb{R}^{B(U(\mathbf{D}), \Omega, \mathcal{G})} &\rightarrow \mathbb{R}^{B(U(\mathbf{D}), \Omega, \mathcal{G})} \\ I &\mapsto \beta I_{+1} \left(\frac{1}{\beta} I(\xi^1) \right) \quad \text{for all } \xi \in \mathbb{R}^{B(U(\mathbf{D}), \Omega, \mathcal{G})}. \end{aligned}$$

First observe that because of translation invariance of I_{+1} , we have

$$T(I + k) \leq T(I) + \beta k$$

for all $k \in U(\mathbf{D})$ and all $I \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$.¹⁵ Thus, following the same steps of Blackwell's contraction Theorem (Blackwell (1965)), it follows that T is a contraction mapping when restricted to $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$ with respect to the metric d_∞ , i.e.,

$$d_\infty(T(I), T(I')) \leq \beta d_\infty(I, I')$$

for all $I, I' \in B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$.

Moreover, notice that T is a self-map when restricted to $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$. To see this, first observe that $T(I)$ is continuous whenever I is continuous. Further, notice that if $\xi \geq \xi'$, then $\xi^1 \geq \xi'^1$ and hence, given that I_{+1} and I are monotone

$$\beta I_{+1} \left(\frac{1}{\beta} I(\xi^1) \right) \geq \beta I_{+1} \left(\frac{1}{\beta} I(\xi'^1) \right).$$

¹⁵To be completely rigorous notice that there exists a unique translation invariant, monotone, and normalized extension of I_{+1} to the tube $B(U(\mathbf{D}), \Omega, \mathcal{G}) + \mathbb{R}$. Thus, the reader can think of I_{+1} as this unique extension to verify the Blackwell discounting condition.

Let $k \in U(\mathbf{D})$, then $k^1 = k$ and hence

$$\beta I_{+1} \left(\frac{1}{\beta} I(k^1) \right) = \beta I_{+1} \left(\frac{k}{\beta} \right) = k$$

thus $T(I)(k) = k$. To verify translation invariance, suppose $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ and $k \in U(\mathbf{D})$ with $\xi + k \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. Then, $(\xi + k)^1 = \xi^1 + k$ and since I_{+1} and I are translation invariant, we have,

$$\begin{aligned} \beta I_{+1} \left(\frac{1}{\beta} I(\xi^1 + k) \right) &= \beta I_{+1} \left(\frac{1}{\beta} I(\xi^1) + \frac{k}{\beta} \right) \\ &= \beta I_{+1} \left(\frac{1}{\beta} I(\xi^1) \right) + k. \end{aligned}$$

Thus, $T(I)(\xi + k) = T(I)(\xi) + k$. Therefore, we have that T is indeed a self-map when restricted to $B_{ce}(U(\mathbf{D}), \Omega, \mathcal{G})$. Hence the result follows by applying the contraction mapping theorem. \square

Generalized rectangularity, MEU, and variational preferences

Generalized rectangularity and MEU

Proof of Corollary 2. Suppose that I_0 and I_{+1} satisfy (3). Then, since both I_0 and I_{+1} are positively homogeneous, it follows that generalized rectangularity is equivalent to

$$\begin{aligned} \min_{P \in \mathcal{P}} \mathbb{E}_P [\xi] &= I_0(\xi) = I_{+1} \left(I_0(\xi^1) \right) \\ &= \min_{\ell \in \mathcal{L}} \mathbb{E}_\ell \left[\min_{P \in \mathcal{P}} \mathbb{E}_P [\xi^1] \right] = \min_{\ell \in \mathcal{L}} \sum_{s \in \mathcal{S}} \ell(s) \min_{P^s \in \mathcal{P}} \mathbb{E}_{P^s} [\xi^1] \end{aligned}$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. This last condition is equivalent to the \mathcal{L} -rectangularity of \mathcal{P} .¹⁶ \square

Generalized rectangularity and variational preferences

Proof of Corollary 3. By Theorem 2 there exist I_0 and I_{+1} that satisfy translation invariance. Observe that by (6) I_{+1} and I_0 are quasi-concave, and so by Lemma

¹⁶The proof of such equivalence is standard. The reader can consult Amarante and Siniscalchi (2019). We can make it available upon request.

25 in Maccheroni et al. (2006a) they are both concave. By Theorem 3 in the same paper we obtain the desired variational representation. In particular, there exist $c_0 : \Delta(\Omega) \rightarrow [0, \infty]$ and $c_{+1} : \Delta(S) \rightarrow [0, \infty]$ such that

$$I_0 = \min_{P \in \Delta(\Omega)} \{ \mathbb{E}_P[\cdot] + c_0(P) \} \text{ and } I_{+1} = \min_{\ell \in \Delta(S)} \{ \mathbb{E}_\ell[\cdot] + c_{+1}(\ell) \}.$$

To prove that the no-gain condition implies generalized rectangularity (3), fix $\xi \in B_0(U(\mathbf{D}), \Omega, \mathcal{G})$ with $\xi = \sum_{i=1}^n \xi_i 1_{A_i}$ for some \mathcal{G} -measurable partition $(A_i)_{i=1}^n$ and $(\xi_i)_{i=1}^n \in U(\mathbf{D})_+^n$. Then, we have that

$$\begin{aligned} \beta I_{+1} \left(\frac{1}{\beta} I_0(\xi^1) \right) &= \beta \min_{\ell \in \Delta(S)} \left\{ \sum_{s \in S} \ell(s) \min_{P_s \in \Delta(\Omega)} \left\{ \sum_{i=1}^n P_s(A_{i,s}) \frac{\xi_i}{\beta} + \frac{1}{\beta} c_0(P_s) \right\} + c_{+1}(\ell) \right\} \\ &= \min_{\ell \in \Delta(S)} \left\{ \sum_{s \in S} \ell(s) \min_{P_s \in \Delta(\Omega)} \left\{ \sum_{i=1}^n P_s(A_{i,s}) \xi_i + c_0(P_s) \right\} + \beta c_{+1}(\ell) \right\} \\ &= \min_{\ell \in \Delta(S)} \min_{(P_s)_{s \in S} \in (\Delta(\Omega))^S} \left\{ \sum_{s \in S} \sum_{i=1}^n \ell(s) P_s(A_{i,s}) \xi_i + \sum_{s \in S} \ell(s) c_0(P_s) + \beta c_{+1}(\ell) \right\} \\ &= \min_{P \in \Delta(\Omega)} \left\{ \sum_{s \in S} \sum_{i=1}^n P_{+1}(s) P_s(A_{i,s}) \xi_i + \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1}) \right\} \\ &= \min_{P \in \Delta(\Omega)} \left\{ \sum_{i=1}^n P(A_i) \xi_i + c_0(P) \right\} \\ &= I_0(\xi). \end{aligned}$$

where the last equality is implied by (7). Generalized rectangularity (3) for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ then holds by continuity and monotone convergence theorem. \square

The following proposition highlights a further connection between our generalized rectangularity and the no-gain condition.

Proposition 3. *Suppose that for all $P \in \Delta(\Omega)$,*

$$c_0(P) = \sup_{h \in \mathbf{H}} \left\{ U(d^h) - \mathbb{E}_P \left[\sum_{t=0}^{\infty} \beta^t u(h_t) \right] \right\}.$$

Then generalized rectangularity (3) implies the inequality

$$c_0(P) \leq \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1})$$

for all $P \in \Delta(\Omega)$. Conversely, (7) implies generalized rectangularity (3).

Proof. We have that

$$\begin{aligned} \min_{P \in \Delta(\Omega)} \left\{ \sum_{s \in S} \sum_{i=1}^n P_{+1}(s) P_s(A_{i,s}) \xi_i + \sum_{s \in S} P_{+1}(s) c_{+1}(P_s) + \beta c_{+1}(\ell) \right\} \\ = \min_{P \in \Delta(\Omega)} \left\{ \sum_{i=1}^n P(A_i) \xi_i + c_0(P) \right\}, \end{aligned}$$

for all $\xi \in B_0(U(\mathbf{D}), \Omega, \mathcal{G})$. By Theorem 3 in Maccheroni et al. (2006a) we have that

$$c_0(P) \leq \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1}).$$

The other side of the claim was already shown in the proof of Corollary 3. \square

References

- Amarante, M. and Siniscalchi, M. (2019). Recursive maxmin preferences and rectangular priors: a simple proof. *Economic Theory Bulletin*, 7(1):125–129.
- Anderson, E. W. (2005). The dynamics of risk-sensitive allocations. *Journal of Economic theory*, 125(2):93–150.
- Baillon, A. and Placido, L. (2019). Testing constant absolute and relative ambiguity aversion. *Journal of Economic Theory*, 181:309–332.
- Bastianello, L. and Faro, J. H. (2022). Choquet expected discounted utility. *Economic Theory*, pages 1–28.
- Berger, L. and Bosetti, V. (2020). Are policymakers ambiguity averse? *The Economic Journal*, 130(626):331–355.
- Blackwell, D. (1965). Discounted dynamic programming. *The Annals of Mathematical Statistics*, 36(1):226–235.
- Bommier, A., Kochov, A., and Le Grand, F. (2017). On monotone recursive preferences. *Econometrica*, 85(5):1433–1466.
- Cai, Y. and Lontzek, T. S. (2019). The social cost of carbon with economic and climate risks. *Journal of Political Economy*, 127(6):2684–2734.

- Cerreia-Vioglio, S., Corrao, R., and Lanzani, G. (2022). (un-)common preferences, ambiguity, and coordination. Technical report, Mimeo.
- Cerreia-Vioglio, S., Maccheroni, F., and Marinacci, M. (2019). Ambiguity aversion and wealth effects. *Journal of Economic Theory*, page 104898.
- Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., and Montrucchio, L. (2011). Uncertainty averse preferences. *Journal of Economic Theory*, 146(4):1275–1330.
- de Castro, L. and Galvao, A. F. (2022). Static and dynamic quantile preferences. *Economic Theory*, 73(2):747–779.
- Dominiak, A. (2013). Iterated choquet expectations: A possibility result. *Economics Letters*, 120(2):155–159.
- Epstein, L. G. (1992). *Behaviour under risk: recent developments in theory and applications*, in “*Advances in Economic Theory: Sixth World Congress*”(JJ Laffont, Ed.), *Econometric Society Monograph*. Cambridge University Press, Cambridge.
- Epstein, L. G. and Schneider, M. (2003a). IID: independently and indistinguishably distributed. *Journal of Economic Theory*, 113(1):32–50.
- Epstein, L. G. and Schneider, M. (2003b). Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31.
- Epstein, L. G. and Zin, S. E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica: Journal of the Econometric Society*, pages 937–969.
- Epstein, L. G. and Zin, S. E. (1991). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of political Economy*, 99(2):263–286.
- Ghirardato, P., Maccheroni, F., and Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118(2):133–173.
- Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of mathematical economics*, 18(2):141–153.

- Grant, S. and Polak, B. (2013). Mean-dispersion preferences and constant absolute uncertainty aversion. *Journal of Economic Theory*, 148(4):1361–1398.
- Hanany, E. and Klibanoff, P. (2009). Updating ambiguity averse preferences. *BE Journal of Theoretical Economics*, 9(1):37.
- Hansen, L. P., Sargent, T. J., and Tallarini, T. D. (1999). Robust permanent income and pricing. *Review of Economic studies*, pages 873–907.
- Herstein, I. N. and Milnor, J. (1953). An axiomatic approach to measurable utility. *Econometrica*, 21:291–297.
- Klibanoff, P. and Hanany, E. (2007). Updating preferences with multiple priors. *Theoretical Economics*, 2(3):261–298.
- Kochov, A. (2015). Time and no lotteries: An axiomatization of maxmin expected utility. *Econometrica*, 83(1):239–262.
- Kochov, A. and Song, Y. (2021). Intertemporal hedging and trade in repeated games with recursive utility. *Available at SSRN*. https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2714337.
- Koopmans, T. C. (1972). Representation of preference orderings over time. *Decision and organization*, 57:100.
- Li, C., Rohde, K. I., and Wakker, P. P. (2023). The deceptive beauty of monotonicity, and the million-dollar question: row-first or column-first aggregation? Working Paper, Erasmus School of Economics.
- Maccheroni, F., Marinacci, M., and Rustichini, A. (2006a). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498.
- Maccheroni, F., Marinacci, M., and Rustichini, A. (2006b). Dynamic variational preferences. *Journal of Economic Theory*, 128(1):4–44.
- Pahlke, M. (2022). Dynamic consistency and ambiguous communication. Working Paper, Paris School of Economics.

- Pires, C. (2002). A rule for updating ambiguous beliefs. *Theory and Decision*, 53(2):137–152.
- Sarver, T. (2018). Dynamic mixture-averse preferences. *Econometrica*, 86(4):1347–1382.
- Savochkin, A., Shklyaev, A., Galatenko, A., et al. (2022). Dynamic consistency and rectangularity for the smooth ambiguity model. Working Paper, New Economic School.
- Strzalecki, T. (2011). Axiomatic foundations of multiplier preferences. *Econometrica*, 79(1):47–73.
- Strzalecki, T. (2013). Temporal resolution of uncertainty and recursive models of ambiguity aversion. *Econometrica*, 81(3):1039–1074.
- Tallarini, T. D. (2000). Risk-sensitive real business cycles. *Journal of monetary Economics*, 45(3):507–532.
- Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of monetary economics*, 24(3):401–421.
- Xue, J. (2020). Preferences with changing ambiguity aversion. *Economic Theory*, 69:1–60.
- Zimper, A. (2011). Re-examining the law of iterated expectations for choquet decision makers. *Theory and decision*, 71(4):669–677.