Recursive Preferences and Ambiguity Attitudes*

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Abstract

We illustrate the strong implications of recursivity, a standard assumption in dynamic environments, on attitudes toward uncertainty. We show that in intertemporal consumption choice problems, recursivity always implies constant absolute ambiguity aversion (CAAA) when applying the standard dynamic extension of monotonicity. Our analysis yields a functional equation called "generalized rectangularity," as it generalizes the standard notion of rectangularity for recursive multiple priors. Our results highlight that if uncertainty aversion is modeled as a form of convexity, recursivity limits us to recursive variational preferences. We propose a novel notion of monotonicity that enables us to overcome this limitation.

Keywords: Dynamic choice, recursive utility, uncertainty aversion, absolute attitudes, generalized rectangularity.

JEL classification: C61, D81.

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1 Introduction

Recursive preferences are a key tool for dynamic economic models. They are the workhorse in macroeconomics and finance for studying a variety of different problems, ranging from consumption-based asset pricing (Epstein and Zin, 1989, Epstein and Zin, 1991), precautionary savings (Weil, 1989, Hansen et al., 1999), business cycles (Tallarini, 2000), Anderson, 2005), and climate change (Cai and Lontzek, 2019). In dynamic models with strategic interaction, they have recently been used to study repeated games (Kochov and Song, 2021) and Bayesian Persuasion (Pahlke, 2022).

Recursivity entails several restrictions on dynamic choice behavior, among which one of the key assumptions is a notion of time or dynamic consistency, i.e., at every time period the decision maker will carry out the plan of actions that was determined to be optimal ex-ante. The assumption of recursivity provides analytical tractability in that it permits the use of well-known tools from dynamic programming.

In this paper, we show that recursivity has strong restrictions on attitudes toward uncertainty, i.e., how uncertainty attitudes change when individuals become better off overall. We focus on a major class of dynamic choice problems, which we refer to as intertemporal consumption choice problems. These problems take place over long horizons, and the source of utility is a consumption stream. Figure 1 offers a graphical representation of a (stochastic) consumption stream. At every period, a shock $s \in S$ is realized, and the total sequence of shocks determines the consumption level at any given time period.

For such problems, the implications of recursivity for ambiguity attitudes depend on the notion of monotonicity. Under the standard notion of monotonicity adopted in the literature (see e.g., Epstein and Schneider, 2003b and Maccheroni et al., 2006b), our first major result (Theorem 1 and Corollary 1) shows that recursive preferences *always* satisfy constant absolute ambiguity aversion (CAAA). From a practical perspective, this fact implies that the generalized recursive smooth ambiguity preferences, studied by Hayashi and Miao (2011) and Ju and Miao (2012) in the context of asset pricing, do no satisfy monotonicity.

As a byproduct, we obtain a generalized "rectangularity" condition for recursive preferences that satisfy the standard notion of monotonicity. Similarly to rectangularity from Epstein and Schneider (2003a, 2003b), generalized rectangularity characterizes "generalized beliefs"—as modeled by certainty equivalents—that are dynamically consis-

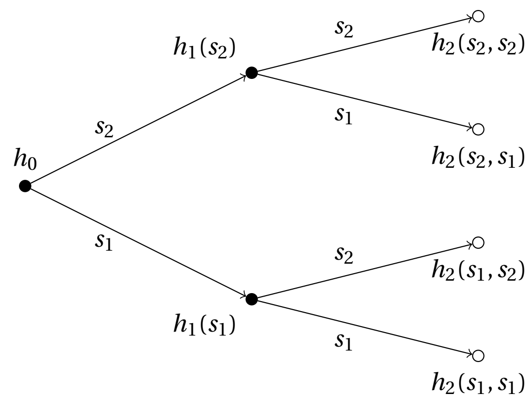

Figure 1: Illustration of a consumption program.

tent. Our generalized rectangularity condition for ex-ante and one-step-ahead certainty equivalents is the following:

$$I_0(\xi) = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right), \tag{1}$$

for all random variable ξ, where I_0 and I_{+1} are certainty equivalents reflecting ex-ante and one-step-ahead beliefs, ξ^1 denotes a shift operator applied to the random variable ξ, and $\beta \in (0,1)$ is a discount factor. The ex-ante certainty equivalent I_0 represents the decision maker's *static preferences*, and therefore contains information about ambiguity attitudes over lifetime utility. This condition unifies existing rectangularity conditions (see Corollaries 2 and 3), and it is applicable to all models of decisions under uncertainty satisfying CAAA.

Condition (1) highlights that in order to elicit the ex-ante certainty equivalent I_0—which reflects, inter alia, beliefs about the entire realizations of shocks—it is enough to elicit beliefs about the shock in the following period. Our Theorem 2 provides the formalization of the previous observation: given the one-step-ahead certainty equivalent I_{+1} generalized rectangularity allows the analyst to recover I_0.[1] Notably, Theorem 2 permits computing I_0 using numerical techniques, even in instances where a closed-form cannot be found, such as when I_{+1} is a Choquet integral.

A key implication of Theorem 1 is that if one assumes that monotone preferences, on top of recursivity, satisfy the notion of uncertainty aversion introduced by Gilboa and Schmeidler (1989) we have that preferences must admit a variational representation in

[1] If the functional equation does not admit an analytic solution, I_0 can be determined by means of the numerical methods we develop.

the sense of Maccheroni et al. (2006a). More specifically, following Cerreia-Vioglio et al. (2011), suppose that there are a utility index u and a quasiconvex function G such that DM's preferences admit the recursive representation

$$V(h) = u(h_0) + \beta \inf_{\ell \in \triangle(S)} G\left(\sum_{s \in S} \ell(s)\left[V \circ h^{s,1}\right], \ell\right).$$

If such preferences satisfy the classical notion of monotonicity, Corollary 2 shows that the aggregator G has the "variational" form

$$G(t, \ell) = t + c(\ell),$$

where $c : \triangle(S) \to [0, \infty]$ is a cost function.

The assumption that a decision maker's attitudes toward uncertainty remain unchanged as one becomes better off contrasts with both introspection and existing experimental literature (see e.g., Baillon and Placido, 2019).[2] Therefore, under tractability—as reflected by recursivity—our results suggest, broadly speaking, a modeling trade-off between. This trade-off is between maintaining monotonicity and accommodating plausible attitudes toward ambiguity, such as non-trivial decreasing absolute ambiguity aversion.

To elucidate why monotonicity leads to such strong conclusions, we introduce a weaker notion of monotonicity for intertemporal consumption choice problems. This axiom, which we refer to as *state-time monotonicity* (Axiom I.9), is a basic consistency principle which requires that an uncertain consumption plan is preferred to another, whenever such a ranking holds jointly at any possible state of the world and at any possible time period. We provide an axiomatization of recursive preferences based on this notion of monotonicity. Specifically, we demonstrate that state-time monotonicity, coupled with an additional more technical condition, implies the standard notion of dynamic consistency.

Finally, building on Epstein (1992), we also consider a different major formalization of a dynamic choice problem: sequential choice problems. Sequential choice problems take place over short intervals of time during which consumption plans can be taken to be fixed, and the source of utility is terminal wealth as opposed to the latter case in which it is given by a consumption stream. Sequential choice problems are typically employed when dealing with updating rules (see e.g., Pires, 2002 and Klibanoff and Hanany, 2007).

[2]As stated by Baillon and Placido (2019) at p. 325: "Our findings seem to encourage the use of ambiguity models that are flexible enough to accommodate changes in ambiguity attitudes at increased utility levels."

In this setting, the implications of recursivity are more nuanced. Savochkin et al. (2022), inter alia, show that CAAA is implied by recursivity when ex-ante preferences admit a smooth ambiguity representation. We show that in general recursivity imposes no restriction on uncertainty attitudes for sequential choice problems. We provide an example of recursive preferences that can allow for unrestricted uncertainty attitudes.

Related literature. Kochov (2015) axiomatizes an intertemporal version of MEU preferences. His results are substantially different from ours. In his setting, translation invariance is implied by an axiom of stationarity which implies a form of indifference to the timing of resolution of uncertainty. In contrast, in our setting, it is monotonicity that implies translation invariance. For more details, see the discussion in Section 3.

Bommier et al. (2017) is the theoretical work closest to the present paper. Similar to our approach, they examine recursive preferences that satisfy state monotonicity, which they refer to just as monotonicity. However, there are notable differences between their study and ours. They investigate the implications of monotonicity for the certainty equivalent I_{+1} alone. In our paper, we investigate the implications of monotonicity for the ex-ante certainty equivalent I_0, which allows us to draw conclusions about general ambiguity attitudes (see also the discussion in Sections 2.2 and 3). Proving that I_0 is translation invariant requires different methods; the techniques from Bommier et al. (2017) based on functional equations to derive translation invariance of the one-step-ahead certainty equivalent I_{+1} cannot be directly applied.

We consider how ambiguity attitudes evolve when a decision maker becomes better off in terms of utility. Using this very notion, Baillon and Placido (2019) and Berger and Bosetti (2020) provide experimental evidence on non-constant ambiguity aversion. In particular, Baillon and Placido's results call for the use of ambiguity models that can accommodate decreasing aversion toward ambiguity. In the context of intertemporal consumption choice problems, we propose a novel notion of monotonicity to address this point.

Alternatively, one may want to predict changes in ambiguity attitude when the decision maker becomes better off in terms of wealth. This approach requires accounting for risk attitudes, as shown by Cerreia-Vioglio et al. (2019). We leave this to future research. For the moment, we observe that our results still apply using Cerreia-Vioglio et al.'s methodology under the assumption of risk neutrality.

Savochkin et al. (2022) consider a setting of sequential choice and characterize recursive smooth ambiguity preferences. While the major source of appeal of the smooth ambiguity model is that it need not satisfy constant ambiguity aversion (either absolute or relative), they show that CAAA is necessary under the assumption of recursivity. Further, they derive a condition for the decision maker's beliefs that ensures recursivity.

In a related work, Li et al. (2023) investigate various forms of monotonicity when preferences are defined over matrices. Their notion of outcome monotonicity is analogous to our concept of state-time monotonicity. They demonstrate that the various forms of monotonicity they consider hold jointly if and only if preferences can be represented by discounted expected utility.

Organization of the paper. Section 2 introduces the notation and the main choice-theoretic objects used in the paper. We consider an infinite horizon version of Epstein and Schneider (2003a) and Strzalecki's (2013) setting. Readers familiar with this setting may wish to skip directly to Section 3, which discusses the main results for intertemporal consumption and sequential choice problems. Section 4 concludes with a discussion of the results in light of the existing literature. All the proofs can be found in the Appendix.

2 Framework

Static choice problems and mathematical preliminaries. Let Ω be a nonempty set of *states of the world* and $\mathscr{G}$ an algebra of *events* over it. By X we denote a convex subset of a vector space, interpreted as a set of *consequences*. A function $f : \Omega \to X$ is said to be a (simple) *act* if it is $\mathscr{G}$-measurable and $f(\Omega)$ is finite; the set of acts is denoted by $\mathbf{F}$. As usual we identify X as a subset of $\mathbf{F}$. We denote by $\succsim$ a binary relation over $\mathbf{F}$, by $\sim$ and $\succ$ its symmetric and asymmetric parts, respectively. A function $V : \mathbf{F} \to \mathbb{R}$ *represents* $\succsim$ if

$$f \succsim g \iff V(f) \geq V(g)$$

for all $f, g \in \mathbf{F}$.

Fix $K \subseteq \mathbb{R}$, we denote by $B_0(K, \Omega, \mathscr{G})$ the set of bounded simple $\mathscr{G}$-measurable functions taking values in K and by $B(K, \Omega, \mathscr{G})$ its closure in the supremum norm $\|\cdot\|_\infty$. We set $B_0(\Omega, \mathscr{G}) := B_0(\mathbb{R}, \Omega, \mathscr{G})$ and $B(\Omega, \mathscr{G}) := B(\mathbb{R}, \Omega, \mathscr{G})$. For all $A \in \mathscr{G}$, we denote by $\mathbf{1}_A$ its indicator function and we identify constant functions as constants. Fix $K \subseteq \mathbb{R}$. A functional $I : B(K, \Omega, \mathscr{G}) \to \mathbb{R}$ is said to be *continuous* if $\lim_{n\to\infty} I(\xi_n) = I(\xi)$ for all sequence $(\xi_n)_{n\geq 0}$ converging pointwise to ξ, and *normalized* if $I(k) = k$ for all $k \in K$. We say that

I is *monotone* if $I(\xi) \geq I\left(\xi'\right)$ whenever $\xi \geq \xi'$, for all $\xi, \xi' \in B(K,\Omega,\mathscr{G})$. If I is continuous, monotone, and normalized, then it is said to be a *certainty equivalent*. Moreover, we say that I is *translation invariant* if $I(\xi + k) = I(\xi) + k$, for all $\xi \in B(K,\Omega,\mathscr{G})$ and $k \in \mathbb{R}$ such that $\xi + k \in B(K,\Omega,\mathscr{G})$.

Fix a nonempty set Y and an algebra $\mathscr{A}$ over it. We denote by $\triangle(Y)$ the set of finitely additive probability measures on Y. If $\mathscr{A}$ is a σ-algebra, we denote by $\triangle^{\sigma}(Y)$ the set of countably additive probability measures on Y. For all nonempty set A, we will denote by $A^{\infty} := \prod_{t\geq 1} A^t$ its countably infinite Cartesian product and by 2^A its power set.

Ambiguity attitudes. In the next sections, we will focus on dynamic choice problems. Building upon the work of Bommier et al. (2017) we will show how the combination of the classical notions of *monotonicity* and *stationarity* of preferences impose restrictions on the decision maker's attitudes toward uncertainty. Before going into the dynamic setting, we discuss the notion of *constant absolute ambiguity aversion* as introduced by Grant and Polak (2013).

Definition 1. *A binary relation $\succsim$ on* $\mathbf{F}$ *exhibits* constant absolute ambiguity aversion (CAAA) *if for all $f \in \mathbf{F}$, $x, y, z \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha y + (1-\alpha)x \Longrightarrow \alpha f + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

In words, constant absolute ambiguity aversion requires that whenever an uncertain alternative is preferred to a sure outcome, "adding" the same certain alternative to both does not invert the preference. Absolute ambiguity attitudes have been thoroughly studied by Xue (2020) and Cerreia-Vioglio et al. (2019) in terms of utility and wealth, respectively. In the context of rational preferences, constant absolute ambiguity aversion is the same as requiring the certainty equivalent being translation invariant.

Proposition 1. *Let $\succsim$ be a binary relation on* $\mathbf{F}$. *Suppose that there exist an affine and non-constant function $u : X \to \mathbb{R}$ and a certainty equivalent $I : B_0(u(X),\Omega,\mathscr{G}) \to \mathbb{R}$ such that $f \mapsto I(u(f))$ represents $\succsim$. The following are equivalent:*

(i) $\succsim$ *satisfies constant absolute ambiguity aversion.*

(ii) I *is translation invariant.*

Proof. The proof is routine and hence it is omitted; the reader can consult Xue (2020). □

This proposition highlights how the constant absolute ambiguity aversion of preferences is translated to the representing certainty equivalent. Many notable models of decision-making under uncertainty satisfy constant absolute ambiguity aversion. Among others, maxmin (Gilboa and Schmeidler, 1989), α-maxmin (Ghirardato et al., 2004), and variational (Maccheroni et al., 2006a) models exhibit constant absolute ambiguity aversion. In their experimental work Baillon and Placido, 2019 provide experimental evidence calling for the use of ambiguity models that can accommodate decreasing aversion toward ambiguity, rather than constant absolute ambiguity aversion.

2.1 Intertemporal consumption choice problems

Our focus will be on dynamic choice problems. Here, we formally present the setting we adopt which is analogous to Strzalecki (2013), but with an infinite horizon (see also Section A.1 of Bommier et al., 2017). More specifically, we work in a stationary IID ambiguity setting, as the one introduced by Epstein and Schneider (2003a).

Let S be a finite set representing the states of the world to be realized in each period.[3] We assume that S has at least three elements and that $\Sigma := 2^S$ is the associated algebra of events. The full state space is denoted by $\Omega := S^\infty$, with a state $\omega \in \Omega$ specifying a complete history $(s_1, s_2, \ldots)$. In each period $t > 0$, the individual knows the partial history $s^t := (s_1, \ldots, s_t)$. The evolution of such information is assumed to be represented by the filtration $(\mathscr{G}_t)_{t\geq 0}$ on Ω where $\mathscr{G}_0 := \{\emptyset, \Omega\}$ and $\mathscr{G}_t := \Sigma^t \times \{\emptyset, S\}^\infty$ for all $t > 0$. Let $\mathscr{G} = \sigma\left(\bigcup_{t\geq 0} \mathscr{G}_t\right)$, that is, the smallest sigma-algebra generated by the union of the sigma-algebras of the filtration $(\mathscr{G}_t)_{t\geq 0}$. The relevant measurable space is $(\Omega, \mathscr{G})$.

The set of possible consumption levels is a compact metrizable space C with at least two distinct elements. The entire consumption set is given by the set of lotteries over C, that is $X = \triangle^\sigma(C)$ which we endow with the topology of weak convergence and the associated Borel sigma-algebra. We identify C as a subset of X, looking at its elements as degenerate lotteries. A *consumption plan* is an X-valued, $(\mathscr{G}_t)_{t\geq 0}$-adapted stochastic process, that is, a sequence $h = (h_t)_{t\geq 0}$ such that $h_t : \Omega \to X$ is $\mathscr{G}_t$-measurable for all $t \geq 0$. The set of all consumption plans is denoted by $\mathbf{H}$ and it is endowed with the product topology (i.e., topology of pointwise convergence). We denote by $\mathbf{D} := X^\infty$ the set of all deterministic consumption plans and identify X as a subset of $\mathbf{D}$ where each $x \in X$ is seen as the constant consumption plan that yields the lottery x in each period.

[3] In finance and macroeconomics S is also interpreted as a set of *shocks*.

For all consumption plans $h \in \mathbf{H}$ and $s \in S$ define the *conditional consumption plan* $h^s \in \mathbf{H}$ by

$$h^s(s_1, s_2, \ldots) = h(s, s_2, \ldots) = (h_0, h_1(s, s_2, \ldots), \ldots)$$

for all $(s_1, s_2, \ldots) \in \Omega$. In words, given a consumption plan $h \in \mathbf{H}$ and a state $s \in S$, the conditional consumption plan h^s is the consumption plan obtained from h when the decision maker knows that in the first period s is realized.

We can use conditional consumption plans to define the continuation of consumption plans. Given $h = (h_0, h_1, h_2, \ldots) \in \mathbf{H}$ and $s \in S$, the *continuation of* h, denoted by $h^{s,1}$, is defined as

$$h^{s,1}(s_1, s_2, \ldots) = (h_1(s, s_2, \ldots), h_2(s, s_2, \ldots), \ldots)$$

for all $(s_1, s_2, \ldots) \in \Omega$. The continuation act $h^{s,1}$ is the consumption plan forwarding h by one period and knowing that in the first period state $s \in S$ realized. For all lotteries $x \in X$ and consumption plans $h \in \mathbf{H}$ we define the concatenation (x, h) as

$$(x, h)(s_1, s_2, \ldots) = (x, h(s_2, s_3, \ldots)) \tag{2}$$

for all $(s_1, s_2, \ldots) \in \Omega$. Likewise, let $s \in S$ and $\xi \in B(K, \Omega, \mathscr{G})$ with $K \subseteq \mathbb{R}$, define $\xi^{s,1} \in B(K, \Omega, \mathscr{G})$ as

$$\xi^{s,1}(s_1, s_2, s_3, \ldots) = \xi(s, s_2, s_3, \ldots)$$

for all $(s_1, s_2, s_3, \ldots) \in \Omega$. For all $h \in \mathbf{H}$, we will denote by h^1 the mapping $s \mapsto h^{s,1}$ and ξ^1 is defined analogously.

Since we will consider preferences that are dynamically consistent, here we only consider ex-ante preferences modeled by a binary relation $\succsim$ on $\mathbf{H}$.[4] Observe that in this setting the set of consumption plans $\mathbf{H}$ can be seen as a subset of acts $\mathbf{F} \subseteq \mathbf{D}^{\Omega}$. Indeed an act here can be seen as a mapping from states into consumption streams

$$\omega \mapsto h(\omega) = (h_0, h_1(\omega), \ldots) \in \mathbf{D}.$$

While this is a straightforward observation, it will be helpful when we will discuss ex-ante representations. It is because of this reformulation of the dynamic problem into a static

[4]Otherwise one would have to state all the axioms for the collection of preferences $(\succsim_{s^t})_{s^t}$ for all possible sequence $s^t \in S^t$, $t \geq 1$. Under the assumption of recursivity, it is not necessary to consider this richer framework. More precisely, if preferences admit the recursive representation given in Definition 2, then it is possible to define conditional preferences $(\succsim_{s^t})_{s^t}$, $t \geq 1$ that satisfy the traditional notion of dynamic consistency.

one that we are able to compare our results with the collected experimental evidence on ambiguity attitudes.

We study a product space and not a general filtration for several reasons. First, it is a standard setting in the decision-theoretic literature (see Strzalecki, 2013 or Bommier et al., 2017). Second, it is the natural setting to study attitudes toward uncertainty. With a general filtration, attitudes toward uncertainty would depend on changing beliefs.[5]

2.2 Recursive preferences

We consider now the intertemporal consumption choice setting described in Section 2.1. The primitive is a binary relation $\succsim$ on the set of consumption plans $\mathbf{H}$. We define preferences $\succsim$ that admit a (time-separable) recursive representation as follows. Given $u : X \to \mathbb{R}$, we define $U : \mathbf{D} \to \mathbb{R}$ as

$$U(d) := \sum_{t \geq 0} \beta^t u(d_t)$$

for all $d \in \mathbf{D}$.

Definition 2. *A preference relation* $\succsim$ *admits a* recursive representation *if there exists a tuple* (V, I_{+1}, u, β) *such that* $V : \mathbf{H} \to \mathbb{R}$ *represents* $\succsim$ *and*

$$V(h) = u(h_0) + \beta I_{+1}\left(V \circ h^1\right), \tag{3}$$

where $u : X \to \mathbb{R}$ *is a continuous function,* $\beta \in (0,1)$*, and* $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$ *is a certainty equivalent.*

The axiomatic characterization of recursive preferences is well understood in the literature (see for example de Castro and Galvao, 2022, Section 4, or Sarver, 2018, Appendix A.1, for similar axiomatizations).

We refer to I_{+1} as a *one-step-ahead certainty equivalent* and to $\succsim$ as a *recursive preference relation.* The one-step-ahead certainty equivalent I_{+1} contains information about uncertainty attitudes restricted to one-step-ahead consumption plans.

Definition 3. *A consumption plan* $h \in \mathbf{H}$ *is said to be* one-step-ahead *if* $h_t = f$ *for some* $\mathscr{G}_1$*-measurable function* $f : \Omega \to X$ *and all* $t \geq 1$.

[5] On this point, see the discussion in Strzalecki (2013) (pp. 1048-1049).

In words, one-step-ahead consumption plans resolve all the uncertainty at $t = 1$ and pay a constant stream of consumption thereafter. Consequently, a priori knowledge of the certainty equivalent I_{+1} is insufficient to describe a decision maker's uncertainty attitudes.

3 Main results

Let $\succsim$ represent the decision maker's preferences on $\mathbf{H}$. We first introduce the following standard axioms.

Axiom I.1 (Weak order)**.** $\succsim$ is complete and transitive.

Axiom I.2 (Continuity)**.** For all $g \in \mathbf{H}$, the sets

$$\{h \in \mathbf{H} : h \succsim g\} \text{ and } \{h \in \mathbf{H} : g \succsim h\}$$

are closed with respect to the product topology.

Axiom I.3 (Nontriviality)**.** There exist $x, y \in X$ such that $x \succ y$.

We consider two classical axioms in the literature on discounting and dynamic choice, namely *time separability* and *stationarity*.

Axiom I.4 (Time separability)**.** For all $x, y, x', y' \in X$ and $d, d' \in \mathbf{D}$, $(x, y, d) \sim (x', y', d)$ if and only if $(x, y, d') \sim (x', y', d')$.

Consider two deterministic consumption plans that yield identical outcomes from the third period onward. Time separability requires that their ranking does not depend on the common continuation. We leave the extension of our results to non-time separable preferences to future research.

Axiom I.5 (Stationarity)**.** For all $x \in X$ and $h, g \in \mathbf{H}$, $h \succsim g$ if and only if $(x, h) \succsim (x, g)$.

Stationarity expresses Koopmans's idea that "the passage of time does not have an effect on preferences." Notice that our notion of stationarity requires not only invariance toward postponing consumption but also delaying the resolution of uncertainty. Indeed, the consumption plan (x, h), as defined in equation (2), is obtained by adding one initial period consumption and postponing the timing of resolution of uncertainty by one period, so that only depends on the information revealed in the second period. Therefore, our notion of stationarity is different from Kochov's (2015) axiom of stationarity. That axiom reflects a notion of preference invariance when changing the timing of consumption, while holding fixed the timing of resolution of uncertainty. In Kochov's setting, this notion implies translation invariance of I_0.[6]

A difference with Bommier et al. (2017) is that we study ambiguity attitudes in a dynamic version of the Anscombe-Aumann framework. Therefore, we require independence over deterministic consumption plans.

Axiom I.6 (Independence for deterministic prospects)**.** For all $d, d', d'' \in \mathbf{D}$ and $\alpha \in (0,1)$,

$$d \sim d' \Longrightarrow \alpha d + (1-\alpha) d'' \sim \alpha d' + (1-\alpha) d''.$$

The next axiom of dynamic consistency, together with the previous ones, characterize recursive preferences with affine u.[7]

Axiom I.7 (Dynamic Consistency)**.** For all $h, g \in \mathbf{H}$ with $h_0 = g_0$,

$$\left[\forall s \in S, h^s \succsim g^s\right] \Longrightarrow h \succsim g.$$

We consider the standard notion of monotonicity typically adopted in the literature (see e.g., Epstein and Schneider, 2003b, Maccheroni et al., 2006b, and Bastianello and Faro, 2022).

Axiom I.8 (State monotonicity)**.** For all $h, g \in \mathbf{H}$,

$$\left[\forall \omega \in \Omega,\ (h_t(\omega))_{t\geq 0} \succsim \left(g_t(\omega)\right)_{t\geq 0}\right] \Longrightarrow h \succsim g.$$

Bommier et al. (2017) show that for recursive preferences, state monotonicity is equivalent to the translation invariance of the one-step-ahead certainty equivalent I_{+1}. However, their characterization is silent about ambiguity attitudes and in particular on the restriction imposed by such an assumption on the ex-ante representation. In our setting, their result can be seen as implying that $\succsim$ has to satisfy constant absolute ambiguity aversion when restricted to one-step-ahead consumption plans.

[6]See also the discussion in Bommier et al. (2017) at pp. 1455-1456.

[7]The standard notion of dynamic consistency requires that if in addition $h^s \succ g^s$ for some $s \in S$, then we have $h \succ g$ (see for example de Castro and Galvao (2022), axiom D3). We consider the weaker notion, as it poses no complications. Our results carry out with the stronger notion with the addition that the certainty equivalents are strictly monotone.

The next result shows that under state monotonicity, $\succsim$ satisfies constant absolute ambiguity aversion. In particular, preferences can be represented by means of an ex-ante certainty equivalent I_0 that is translation invariant. Moreover, I_0 is linked to I_{+1} through an equation which we refer to as generalized rectangularity as we will show that it is a generalization of the rectangularity of MEU preferences from Epstein and Schneider (2003a).

Theorem 1. *Let $\succsim$ be a binary relation on $\mathbf{H}$. The following are equivalent:*

(i) $\succsim$ satisfies I.1-I.7.

(ii) $\succsim$ admits a recursive representation with u affine and non-constant, I_{+1} translation invariant, and there exists a translation invariant certainty equivalent I_0 : $B(U(\mathbf{D}), \Omega, \mathscr{G}) \to \mathbb{R}$ such that $\succsim$ is represented by

$$h \mapsto I_0\left(\sum_{t\geq 0} \beta^t u(h_t)\right).$$

Moreover, I_0 satisfies

$$I_0(\xi) = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right), \tag{4}$$

where $I_0(\xi^1)(s) = I_0\left(\xi^{s,1}\right)$ for all $\xi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$ and $s \in S$.

Outline of the proof. To show the "only if" part, we use state monotonicity and continuity for the existence of a certainty equivalent I_0 such that the mapping

$$h \mapsto I_0\left(\sum_{t\geq 0} \beta^t u(h_t)\right) \tag{5}$$

represents $\succsim$. We show that I_0 must satisfy translation invariance by stationarity. Using translation invariance, (5) and (3), we show that I_0 and I_{+1} are related to each other by (4), which implies that I_{+1} is also translation invariant. The details are in the Appendix. ■

To illustrate, in the following example we provide a preference relation that admits a recursive representation but does not satisfy state monotonicity (Axiom I.8).

Example 1. Assume that $S = \{H, T\}$ and $X = [0, 100]$. Consider recursive preferences with the smooth ambiguity certainty equivalent

$$I_{+1} : \xi \mapsto \phi^{-1}\left(\frac{1}{2}\phi\left(\mathbb{E}_{P^1}[\xi]\right) + \frac{1}{2}\phi\left(\mathbb{E}_{P^2}[\xi]\right)\right),$$

where $P^1(H) = P^1(T) = \frac{1}{2}$ and $P^2(H) = \frac{2}{3} = 1 - P^2(T)$. Assume $\phi : r \mapsto \sqrt{r}$, $u : x \mapsto \mathbb{E}_x[\mathrm{Id}]$, and $\beta = 0.1$. Let $h = (1, f, \ldots, f, \ldots)$ and $g = (0, 10 + f, f, \ldots, f, \ldots)$ where $f : \Omega \to X$ satisfies $f(\omega) = 1$ for all $\omega = (H, s_2, s_3, \ldots)$ and equals 0 otherwise. Observe that

$$U(h(\omega)) = \sum_{t \geq 0} \beta^t h_t(\omega) = \sum_{t \geq 0} \beta^t g_t(\omega) = U(g(\omega)),$$

for all $\omega \in \Omega$. Yet we also have

$$\begin{aligned}
V(h) &= 1 + 0.1 \left(\frac{1}{2} \sqrt{\frac{1}{0.9} \cdot \frac{1}{2}} + \frac{1}{2} \sqrt{\frac{1}{0.9} \cdot \frac{2}{3}} \right)^2 \\
&\approx 1.0645 \\
&< 1.0648 \\
&\approx 0.1 \left(\frac{1}{2} \sqrt{10 + \frac{1}{0.9} \cdot \frac{1}{2}} + \frac{1}{2} \sqrt{10 + \frac{1}{0.9} \cdot \frac{2}{3}} \right)^2 = V(g),
\end{aligned}$$

thus implying a violation of state monotonicity. Notice however that h is neither better nor worse than g according to state-time monotonicity.

A more striking consequence of Theorem 1 is the translation invariance of the ex-ante certainty equivalent I_0. Indeed, given the affinity of u, we observe that recursive preferences under state monotonicity satisfy constant absolute ambiguity aversion.

Corollary 1. *Suppose that $\succsim$ admits the representation (3) and satisfies Axiom I.8. Then $\succsim$ exhibits constant absolute ambiguity aversion.*

Proof. The statement follows by applying Theorem 1 and then Proposition 1 to I_0, exploiting its continuity. ■

Notice that the ambiguity attitudes of the agent are exhibited by the properties of the ex-ante certainty equivalent representation. In particular, any consumption plan $h = (h_0, h_1, \ldots)$ can be seen as an act $h : \Omega \to \mathbf{D}$. Therefore, I_0 is represents *static preferences* over these acts. We can therefore talk about ambiguity attitudes using the lifetime utility of the agent as the basic utility over deterministic alternatives.

Therefore, Theorem 1 and Corollary 1 highlight an important modeling trade-off. If one wants to preserve decreasing absolute ambiguity aversion, monotonicity should be weakened. On the other hand, we also show that under state monotonicity—or better, under constant absolute ambiguity aversion—one obtains a useful condition that we call *generalized rectangularity* (4).

One can think of rectangularity as a generalized form of law of iterated expectations. The primary distinction from a conventional law of iterated expectations is the former's dependence on the degree of impatience. However, this dependence vanishes when the certainty equivalents are positively homogeneous.

This functional equation enables us to solve for the ex-ante certainty equivalent I_0 using the one-step-ahead certainty equivalent I_{+1}. The following result demonstrates that, with a fixed I_{+1}, there exists a unique I_0 that satisfies equation (4). Notably, a standard numerical procedure based on the contraction mapping theorem can be used to find I_0. The interpretation is that given a one-step-ahead certainty equivalent, under recursivity and monotonicity one can recover uniquely the ex-ante certainty equivalent.

In all the results that follow we will always refer to U as the discounted utility function on $\mathbf{D}$ defined through some $\beta \in (0,1)$ and non-constant, affine, and continuous function $u : X \to \mathbb{R}$.

Theorem 2. *Fix a translation invariant certainty equivalent $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$. There exists a unique translation invariant certainty equivalent $I_0^* : B(U(\mathbf{D}), \Omega, \mathscr{G}) \to \mathbb{R}$ such that*

$$I_0^*(\xi) = \beta I_{+1}\left(\frac{1}{\beta} I_0^*\left(\xi^1\right)\right)$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$.

Outline of the proof. Using the fact that I_{+1} is translation invariant, this result follows by applying a version of Blackwell's contraction mapping theorem to a suitably chosen operator. The complete proof is available in the Appendix. ■

In general, however, one should not expect an explicit solution of this functional equation. For example, assume that

$$I_{+1}(\xi) = \int \xi \, d v_{+1} \text{ for all } \xi \in B(U(\mathbf{D}), S, \Sigma),$$

where $v_{+1} : \Sigma \to [0,1]$ is a capacity that is neither convex nor concave.[8] In this case, there need not be a capacity $v_0 : \mathscr{G} \to [0,1]$ such that generalized rectangularity holds, which in this case is equivalent to

$$\int \xi \, d v_0 = \int \int \xi^1 d v_0 d v_{+1},$$

[8]The integral in display is in terms of Choquet. The reader can consult Denneberg (1994) for the formal definitions employed in this section.

for all $\xi \in B(U(\mathbf{D}),\Omega,\mathscr{G})$.[9] Nevertheless, after the proof of Theorem 2, we provide a simple numerical technique that enables an analyst to find I_0 given I_{+1}.[10] For example, in the Choquet case our numerical technique allows one to find a certainty equivalent I_0^* that satisfies

$$I_0^*(\xi) = \int I_0^*\left(\xi^1\right) dv_{+1}.$$

A practical special case of interest is a recursive rank-dependent utility model, in which the certainty equivalent takes the form:

$$I_{+1}(\xi) = \int_{-\infty}^{0} g\left(p\{s:\xi(s)\geq t\}-1\right) + \int_{0}^{\infty} g\left(p\{s:\xi(s)\geq t\}\right) dt.$$

Our result enables to compute I_0 even when the probability distortion function g is neither everywhere convex nor concave.

Implications for uncertainty averse preferences. A major implication is that under state monotonicity, the only recursive preferences that satisfy uncertainty aversion are variational.

Before providing the results we will need some more notation. We recall that $\mathscr{G} = \sigma\left(\bigcup_{t\geq 1}\mathscr{G}_t\right)$. For all $A \in \mathscr{G}$ and $s \in S$, let

$$A_s = \left\{(s_t)_2^\infty : \left(s,(s_t)_2^\infty\right) \in A\right\}.$$

Further, given $P \in \triangle(\Omega)$ and $s \in S$, P_{+1} denotes the marginal over the first coordinate while P_s denotes the marginal over the cylinder set $\{s^\infty \in \Omega : s_1 = s\}$. We say that $c : \triangle(\Omega) \to [0,\infty]$ is *grounded* if its infimum value is zero. In addition, we say that c is a *cost function* if it is convex, grounded, and lower semicontinuous.

Corollary 2. *Suppose that $\succsim$ admits the representation (3) with u affine and non-constant and satisfies*

$$h \sim g \Rightarrow \alpha h + (1-\alpha) g \succsim h, \tag{6}$$

for all $h, g \in \mathbf{H}$ and $\alpha \in (0,1)$. Then $\succsim$ satisfies state monotonicity if and only if there exist cost functions $c_{+1} : \triangle(S) \to [0,\infty]$, $c_0 : \triangle(\Omega) \to [0,\infty]$ such that for all $h \in \mathbf{H}$,

$$V(h) = u(h_0) + \beta \min_{\ell\in\triangle(S)} \left\{\mathbb{E}_\ell\left[V\circ h^1\right] + c_{+1}(\ell)\right\}$$

[9] We refer to Zimper (2011) and Dominiak (2013) for cases in which it does or does not hold.

[10] In terms of numerical implementation, since we rely on the contraction mapping theorem, as shown Rust et al. (2002) there may be a curse of dimensionality that depends on the cardinality of the set S. We refer to that same paper for potential numerical techinques to address this problem.

and

$$I_0\left(\sum_{t\geq 0}\beta^t u(h_t)\right) = \min_{P\in\triangle(\Omega)}\left\{\mathbb{E}_P\left[\sum_{t\geq 0}\beta^t u(h_t)\right] + c_0(P)\right\}.$$

Moreover, a sufficient condition for (4) *is given by*

$$c_0(P) = \sum_{s\in S} P_{+1}(s)c_0(P_s) + \beta c_{+1}(P_{+1}), \tag{7}$$

for all $P \in \triangle(\Omega)$.

Outline of the proof. Because of (6) and state monotonicity, I_{+1} is translation invariant and quasi-concave. Likewise, there exists I_0 that is translation invariant and quasi-concave. Hence by standard results the certainty equivalents I_{+1} and I_0 have the desired variational representation. It then follows that generalized rectangularity is implied by (7). See the Appendix for the full proof. ■

Observe that (7) is reminiscent of the no-gain condition in Maccheroni et al. (2006b). In this setting, the no-gain condition is sufficient only because we do not have unbounded utility. We discuss more technical points related to the necessity of (7) in the Appendix. In particular, under weak technical requirements the inequality

$$c_0(P) \leq \sum_{s\in S} P_{+1}(s)c_0(P_s) + \beta c_{+1}(P_{+1}), \tag{8}$$

is implied by generalized rectangularity. This result further illustrates how our generalized rectangularity subsumes the major characterizations of recursive beliefs.

The next example, based on entropic cost functions, provides a notable case of cost functions that satisfy (7). Fix a measurable space Y. Given $Q \in \triangle^\sigma(Y)$, the relative entropy $R(\cdot\|Q)$ is a mapping from $\triangle(Y)$ into $[0,\infty]$ defined by

$$R(P\|Q) = \begin{cases} \int_Y \left(\log\frac{dP}{dQ}\right)dP, & \text{if } P \in \triangle^\sigma(Y) \text{ and } P \ll Q \\ \infty & \text{otherwise.}\end{cases}$$

Example 2. Given $Q \in \triangle^\sigma(\Omega)$ and $\theta > 0$, let

$$c_0(P) = \frac{1}{\theta}R(P\|Q) \quad \text{for all } Q \in \triangle^\sigma(\Omega),$$

and

$$c_{+1}(\ell) = \frac{1}{\beta\theta}R(\ell\|Q_{+1}) \quad \text{for all } \ell \in \triangle^\sigma(S).$$

This entropic cost formulation corresponds to recursive multiplier preferences (see Strzalecki, 2011 for the static formulation). Condition (7) reduces to

$$R(P\|Q) = \sum_{s\in S} P_{+1}(s) R(P_s\|Q_s) + R(P_{+1}\|Q_{+1}),$$

which can be shown to be always satisfied for all $P \in \triangle(\Omega)$ with the same reasoning used in Maccheroni et al. (2006b) (see their Theorem 3).

3.1 Generalized rectangularity for MEU and variational preferences

In this subsection we show how generalized rectangularity behaves in the case of maxmin and variational preferences. In particular, we show how it is equivalent to the original rectangularity from Epstein and Schneider (2003a), and Epstein and Schneider (2003b) for recursive MEU preferences and its connection with the no-gain condition introduced by Maccheroni et al. (2006b).

For our present specification rectangularity takes the following form. Let $\mathscr{L} \subseteq \triangle(S)$ and $\mathscr{P} \subseteq \triangle(\Omega)$. We say that $\mathscr{P}$ is *$\mathscr{L}$-rectangular* if $P \in \mathscr{L}$ if and only if there exist $\ell \in \mathscr{L}$ and $\{Q^s \in \mathscr{P} : s \in S\}$ such that

$$P(A) = \sum_{s\in S} \ell(s) Q^s(A_s)$$

for all $A \in \mathscr{G}$. This is the classic notion of rectangularity as introduced by Epstein and Schneider (2003a). We provide the following characterization.

Corollary 3 (Rectangularity for MEU[11])**.** *Suppose that $\mathscr{P} \subseteq \triangle(\Omega)$ and $\mathscr{L} \subseteq \triangle(S)$ are convex and weak* compact sets. If*

$$I_0(\xi) = \min_{P\in\mathscr{P}} \mathbb{E}_P[\xi] \quad \textit{for all } \xi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$$

and

$$I_{+1}(\xi) = \min_{\ell\in\mathscr{L}} \mathbb{E}_\ell[\xi] \quad \textit{for all } \xi \in B(U(\mathbf{D}), S, \Sigma),$$

then the following are equivalent

(i) I_0 *and* I_{+1} *satisfy* (4).

[11] See also Epstein and Schneider (2003a) equations 2.4 and 2.6.

(ii) $\mathscr{P}$ and $\mathscr{L}$ satisfy

$$\min_{P\in\mathscr{P}} P(A) = \min_{\ell\in L}\left[\sum_{s\in S}\ell(s)\min_{P\in\mathscr{P}} P(A_s)\right] \quad \textit{for all } A\in\mathscr{G}. \tag{9}$$

(iii) $\mathscr{P}$ is $\mathscr{L}$-rectangular.

Likewise, one can also verify that if I_0 and I_{+1} are positively homogeneous, so that they admit the self-dual representation from Chandrasekher et al. (2022), then generalized rectangularity reduces to the rectangularity condition from that same paper (see Appendix S.2.1 of that paper).

3.2 A weaker form of monotonicity

One may be surprised that state monotonicity has such strong implications for recursive preferences. Here we propose a novel notion of monotonicity for recursive preferences that is weaker than state monotonicity. We show that replacing dynamic consistency with this new, weaker form of monotonicity, coupled with an additional, more technical axiom, effectively characterizes recursive preferences in Definition 3. This result clarifies in what sense state monotonicity is "too strong" of a notion of monotonicity.

Axiom I.9 (State-time monotonicity)**.** For all $h, g \in \mathbf{H}$,

$$\left[\forall\omega\in\Omega, t\geq 0,\ h_t(\omega)\succsim g_t(\omega)\right] \implies h\succsim g.$$

State-time monotonicity can be seen as a minimal consistency requirement. If a decision maker is asked to compare two uncertain consumption streams h and g and h weakly dominates g (according to the DM's preferences) in each period and each state, then h should be preferred to g.

A further interpretation of state-time monotonicity is to think of the *relevant* state space as being the set of all state-time pairs (ω, t). Seeing each of these pairs as a node on our event tree, if the consumption level of h is higher than that of g at each node, then h should be preferred to g. Figure 1 offers a graphical description of state-time monotonicity. The act h pays better than g at every possible node, while h' pays better than g' at every possible node except at the node s_2. By state-time monotonicity h should be preferred to g but h' need not be preferred to g'. State monotonicity's interpretation is analogous to that of state-time monotonicity, but when the *relevant* state space is just Ω.

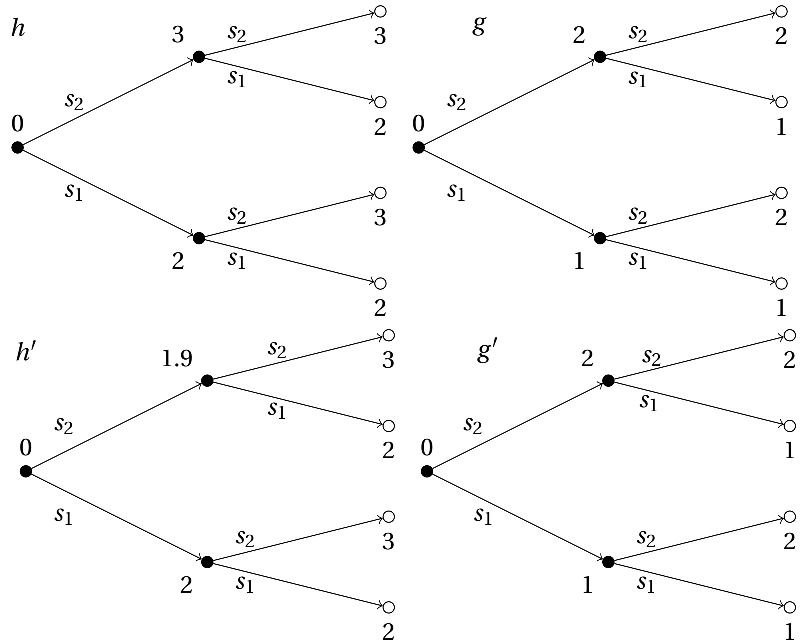

Figure 2: Illustration of state-time monotonicity

Axiom I.10 (One-step-ahead equivalence)**.** For all $h \in \mathbf{H}$, there exists a one-step-ahead consumption plan $h^{+1} \in \mathbf{H}$ such that $h_0 = h_0^{+1}$, $h \sim h^{+1}$, and

$$\left(h^{+1}\right)^s \sim h^s$$

for all $s \in S$.

This axiom requires the existence of a one-step-ahead certainty equivalent. We show how the combination of state-time monotonicity and one-step-ahead equivalence yields *dynamic consistency.*

Theorem 3. *Let $\succsim$ be a binary relation on $\mathbf{H}$. The following are equivalent:*

(i) $\succsim$ satisfies axioms I.1-I.6, I.9, and I.10.

(ii) $\succsim$ admits a recursive representation with affine and non-constant u.

The main feature of this representation theorem it breaks down dynamic consistency (Axiom I.7) into two more fundamental axioms, one of which is a more intuitive requirement of monotonicity (Axiom I.9).[12]

[12] It is important to notice that the independence axiom we impose on deterministic prospects is unnecessary for obtaining a recursive representation. Without this axiom the second point of the statement would read just as: $\succsim$ *admits a recursive representation.*

3.3 Sequential choice

Under Epstein's terminology (see Epstein, 1992), one can distinguish sequential choice problems from intertemporal consumption choice problems. The former setting models situations taking place over short intervals of time during which consumption/savings plans can be taken to be fixed. The source of utility is terminal wealth rather than a consumption sequence. Therefore, in this setting there is no distinction between state monotonicity and state-time monotonicity. The key axiom characterizing a recursive representation in this setting is a version of the sure-thing principle (see the condition in 10). Based on this, there is no reason to expect recursive preferences in this setting to satisfy CAAA.

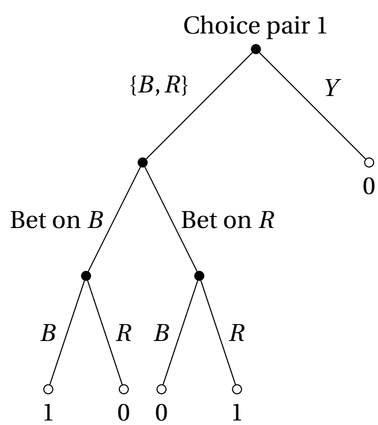

Figure 3: An Ellsberg type sequential choice problem

In sequential choice problems, the decision maker (DM) has to choose a bet ex-ante and at an interim information stage, where the DM can bet based on partial resolution of uncertainty (see for example Section 1.1 in Hanany and Klibanoff, 2009 for an example). Subsequently, a state is realized and payment occurs depending on the DM's choice. Figure 3 offers a graphical representation of a sequential choice problem.

This setting builds upon the one on static choice problems presented in Section 2. In particular, there is a finite partition Π of Ω, and $\mathscr{G} := \sigma(\Pi)$ is the σ-algebra generated by the partition Π. Given $\omega \in \Omega$, we denote by $\Pi(\omega)$ the unique element of Π such that $\omega \in \Pi(\omega)$. The set of acts is denoted by $\mathbf{F}$ and the DM's preferences are expressed as unconditional and conditional complete and transitive relations over $\mathbf{F}$. Such preferences are denoted by $\langle \succsim, (\succsim_E)_{E\in\Pi}\rangle$. The binary relation $\succsim$ models the ex-ante preferences of the DM while $\succsim_E$ models the DM's preferences conditional on the event $E \in \Pi$. In this setting, a recursive representation takes the following form.

Definition 4. *Preferences* $\langle \succsim, (\succsim_E)_{E\in\Pi} \rangle$ *admit a* recursive representation *if there exist an affine function* $u : X \to \mathbb{R}$ *and certainty equivalents* $I : B_0(u(X), \Omega, \mathscr{G}) \to \mathbb{R}$, *and* $\bar{I}(\omega, \cdot) : B_0(u(X), \Omega, \mathscr{G}) \to \mathbb{R}$ *for all* $\omega \in \Omega$ *such that*

1. $I(u(f)) = I(\bar{I}(\cdot, u(f)))$ *for all* $f \in \mathbf{F}$,
2. $\bar{I}\left(\omega, u(f)1_{\Pi(\omega)} + u(g)1_{\Pi(\omega)^c}\right) = \bar{I}(\omega, u(f))$ *for all* $\omega \in \Omega$ *and* $f, g \in \mathbf{F}$,
3. $f \mapsto I(u(f))$ *represents* $\succsim$,
4. *for all* $E \in \Pi$ *and* $\omega \in E$ *the mapping* $f \mapsto \bar{I}(\omega, u(f))$ *represents* $\succsim_E$.[13]

The key axiom for this representation is the following version of sure-thing principle with respect to Π:

$$fEh \succsim gEh \quad \Leftrightarrow \quad fEh' \succsim gEh' \quad \text{for all } f, g, h, h' \in \mathbf{F} \text{ and } E \in \Pi. \tag{10}$$

There is no a priori reason to expect that this axiom should entail restrictions to attitudes toward uncertainty.

However, Savochkin et al. (2022) show that CAAA is implied by recursivity when Π is a binary partition and $\succsim$ belongs to the smooth ambiguity class, i.e., when preferences admit the representation

$$V : f \mapsto \phi^{-1}\left(\int_{\triangle(\Omega)} \phi\left(\int_{\Omega} u(f)\, d\mu\right) d\pi\right),$$

for some concave, differentiable, and strictly increasing function ϕ. Indeed, their results imply that ϕ must be either linear or such that $\phi(x) = -e^{-\theta x}$ for some $\theta > 0$.[14] This result does not however hold more different classes of preferences, as we show in the next example.

[13] An axiomatization of recursive preferences in this setting can be found in Cerreia-Vioglio et al. (2022), Proposition 8.

[14] In their setting, the key behavioral restriction entailing constant absolute ambiguity aversion is the following:

$$[\forall E \in \Pi,\ f \sim_E g] \Rightarrow [\forall E \in \Pi,\ fEh \sim gEh].$$

In their proof, step 3a (p. 23 of their working paper), such a property is used to derive a functional equation that implies that ϕ is an exponential, therefore satisfying CAAA.

Example 3. Fix a strictly increasing function $\phi : u(X) \to \mathbb{R}$. Consider $\succsim$ represented by

$$I(u(f)) = \phi^{-1}\left(\int \phi(u(f))d\mu\right),$$

for all $f \in \mathbf{F}$ and $\succsim_E$ is represented by $I_E(u(f)) = \phi^{-1}\left(\int \phi(u(f))d\mu_E\right)$ for all $E \in \Pi$.[15] Therefore, we can set $\bar{I}(\omega, u(f)) = I_E(u(f))$ whenever $\omega \in E$. It is immediately verifiable that properties 2-4 in Definition 4 above are satisfied. Property 1 is also satisfied since we have

$$\begin{aligned} I(\bar{I}(\cdot, u(f))) &= I\left(\phi^{-1}\left(\int \phi(u(f))d\mu_E\right)\right) \\ &= \phi^{-1}\left(\int\left(\int \phi(u(f))d\mu_E\right)d\mu\right) = \phi^{-1}\left(\int \phi(u(f))d\mu\right), \end{aligned}$$

by the law of iterated expectations. Therefore, the function ϕ can have any shape, thus allowing for arbitrary attitudes toward uncertainty. For example, if $\phi(x) = x^\rho$ for $0 < \rho < 1$, then $\succsim$ will not satisfy CAAA over $\mathbf{F}$.

4 Concluding remarks

Our paper has shown that for preferences that are recursive and hence tractable one cannot have simultaneously

1. the standard notion of monotonicity;
2. (strictly) decreasing absolute ambiguity aversion, as consistent with the experimental literature.

We have provided a full characterization monotone recursive preferences, establishing in particular a generalized notion of rectangularity of beliefs.

At the same time, for intertemporal consumption choice problems—the main focus of the applied literature—we suggest a novel notion of monotonicity, namely state-time monotonicity. State-time monotonicity is compatible with discarding point (1) from the list above and allows for the examination of tractable preferences in a dynamic setting while accommodating realistic uncertainty attitudes.

[15] Here, $\mu_E := \mu(\cdot|E)$.

Appendix

Proofs of the main results

We provide the proofs of our main results in several steps starting with the derivation of discounted utility representation $d \mapsto \sum_{t\geq 0} \beta^t u(d_t)$ for $\succsim$ over the set of deterministic processes $\mathbf{D}$. We detail the proof to make our paper as self-contained and explicit as possible. In particular, we follow the approach of Bastianello and Faro (2022). Consider the following axioms:

(P0) (Weak order) $\succsim$ is complete and transitive.

(P1) (Continuity) For all compact sets $K \subseteq X$ and all $d \in \mathbf{D}$, the sets $\{d' \in K^\infty | d \succsim d'\}$ and $\{d' \in K^\infty | d' \succsim d\}$ are closed in the product topology over K^∞.

(P2) (Sensitivity) There exist $x, y \in X$, $d \in \mathbf{D}$ such that $(x, d) \succ (y, d)$.

(P3) (Stationarity) For all $x \in X$ and $d, d' \in \mathbf{D}$, $d \succsim d'$ if and only if $(x, d) \succsim (x, d')$.

(P4) (Time separability) For all $x, y, x', y' \in X$ and $d, d' \in \mathbf{D}$, $(x, y, d) \sim (x', y', d)$ if and only if $(x, y, d') \sim (x', y', d')$.

(P5) (Monotonicity) Let $d, d' \in \mathbf{D}$. If $d_t \succsim d'_t$ for all $t \geq 0$, then $d \succsim d'$; if moreover $d_t \succ d'_t$ for some $t \geq 0$, then $d \succ d'$.

Proposition 2 (Bastianello and Faro, 2022)**.** *A binary relation $\succsim$ over $\mathbf{D}$ satisfies **(P0)**-**(P5)** if and only if there exists a continuous function $u : X \to \mathbb{R}$ and a discount factor $\beta \in (0,1)$ such that $\succsim$ is represented by*

$$U : d \mapsto \sum_{t\geq 0} \beta^t u(d_t).$$

Proof. See Proposition 5 in Bastianello and Faro (2022). ■

Lemma 1. *Axioms I.1, I.2, I.3, and I.5 imply **(P2)***

Proof. The proof is analogous to Lemma 5 in Kochov (2015); we report it here for the sake of completeness. Suppose that $(x, d) \sim (y, d)$ for all $x, y \in X$ and all $d \in \mathbf{D}$. Then, by stationarity, $(z, x, d) \sim (z', x, d) \sim (z', y, d)$ for all $z, z', x, y \in X$ and $d \in \mathbf{D}$. Repeating this argument we have that $d \sim d'$ for all $d, d' \in \mathbf{D}$ that differ in at most finitely many points. Let $d = (x_0, x_1, \ldots)$ and $d' = (y_0, y_1, \ldots)$ in $\mathbf{D}$ and define $d^t = (x_0, \ldots, x_{t-1}, y_t, y_{t+1}, y_{t+2}, \ldots)$. The previous argument shows that $d^t \sim d'$ for all $t \geq 1$ and $(d^t)_{t\geq 1}$ converges to d. By continuity and completeness of $\succsim$ we have $d \sim d'$. Since $d, d' \in \mathbf{D}$ were chosen arbitrarily, we have a contradiction of Axiom I.3. ■

Lemma 2 (Bastianello and Faro, 2022)**.** *Axioms I.1, I.2, I.4, and I.5 imply **(P5)**.*

Proof. See Lemma A.2 in Bastianello and Faro (2022). ■

Lemma 3. *A binary relation $\succsim$ over $\mathbf{D}$ axioms I.1, and I.2-I.6 if and only if there exists a non-constant, continuous, and affine function $u : X \to \mathbb{R}$ and a discount factor $\beta \in (0,1)$ such that $\succsim$ is represented by*

$$U : d \mapsto \sum_{t \geq 0} \beta^t u(d_t).$$

Proof. By Lemmas 1 and 2 we have that axioms I.1 and I.2-I.6 imply **(P0)**, **(P2)**, and **(P5)**. It is immediately observable that all the others **(P1)**, **(P3)**, **(P4)** are directly implied by axioms I.1 and I.2-I.6. Therefore, by Proposition 2 we have that there exists a continuous function $u : X \to \mathbb{R}$ and a discount factor $\beta \in (0,1)$ such that $U : d \mapsto \sum_{t \geq 0} \beta^t u(d_t)$ represents $\succsim$. Now notice that when restricted to X the preference relation $\succsim$ satisfies all the hypotheses of Theorem 8 in Herstein and Milnor (1953); therefore $\succsim$ admits an affine utility representation v. Since v must be cardinally unique it follows that u must be a positive affine transformation of v and as such it must be affine. ■

Before the proof of the next result we need the following definition.

Definition 5 (t-steps-ahead plan)**.** *A consumption plan $h \in \mathbf{H}$ is said to be t-steps-ahead if $h_\tau = f$ for some $\mathcal{G}_t$-measurable function $f : \Omega \to X$ and all $\tau \geq 1$.*

Let $\mathbf{H}_t$ denote the set of t-steps-ahead plans, for all $t \geq 1$.

Proof of Theorem 3. $[(ii) \Rightarrow (i)]$. Checking that the axioms are necessary for the representation is routine, except for the one-step-ahead equivalence and state-time monotonicity, whose necessity we now show. To do so, take any $h \in \mathbf{H}$ and let $(x^s)_{s \in S} \in X^S$ be such that $U(x^s, x^s, \ldots) = V(h^s)$ for all $s \in S$. Define $h_0^{+1} = h_0$ and $h_t^{+1} = f$ for all $t \geq 1$ where $f : \Omega \to X$ with $f(s) = x^s$ for all $s \in S$. Clearly, f is $\mathcal{G}_1$-measurable. Observe that by construction we have $h_0 = h_0^{+1}$ and $\left(h^{+1}\right)^s \sim h^s$ for all $s \in S$. Since $\succsim$ admits a recursive representation with I_{+1} monotone, it must satisfy

$$\left[\forall s \in S,\ h^s \succsim g^s\right] \implies h \succsim g$$

for all $h, g \in \mathbf{H}$ such that $h_0 = g_0$. Therefore, $\left(h^{+1}\right)^s \sim h^s$ for all $s \in S$ and $h_0 = h_0^{+1}$ imply $h \sim h^{+1}$.

Turning to state-time monotonicitiy, we divide the proof in three steps. Take $h, g \in \mathbf{H}$ such that $h \geq_{STM} g$, that is

$$\forall \omega \in \Omega, t \geq 0, \; h_t(\omega) \succsim g_t(\omega).$$

Step 1. If $h, g \in \mathbf{H}_1$, then $h \succsim g$. Indeed, $V\left(h^{s,1}\right) \geq V\left(g^{s,1}\right)$ for all $s \in S$, so that the statement follows by monotonicity of I_{+1}.

Step 2. If $h, g \in \mathbf{H}_t$, then there exist $\hat{h}, \hat{g} \in \mathbf{H}_{t-1}$ such that $h \sim \hat{h}, g \sim \hat{g}$, and $\hat{h} \geq_{STM} \hat{g}$. Indeed, for all $s^{t-1} = (s_1, \ldots, s_{t-1}) \in S^{t-1}$ with $t \geq 2$, using the recursive representation one can find $x(s_1, \ldots, s_{t-1}), x'(s_1, \ldots, s_{t-1}) \in X$ satisfying

$$(h)^{s^{t-1}, t-1} \sim (x(s_1, \ldots, s_{t-1}), x(s_1, \ldots, s_{t-1}), \ldots),$$

and

$$\left(g\right)^{s^{t-1}, t-1} \sim \left(x'(s_1, \ldots, s_{t-1}), x'(s_1, \ldots, s_{t-1}), \ldots, \right).$$ [16]

Now define $\hat{h}$ and $\hat{g}$ as follows[17]

$$\hat{h}_\tau(s_1, \ldots, s_\tau) = \begin{cases} h_\tau(s_1, \ldots, s_\tau) & \tau \leq t-1, \\ x(s_1, \ldots, s_{t-1}) & \tau > t-1, \end{cases}$$

and

$$\hat{g}_\tau(s_1, \ldots, s_\tau) = \begin{cases} g_\tau(s_1, \ldots, s_\tau) & \tau \leq t-1, \\ x'(s_1, \ldots, s_{t-1}) & \tau > t-1. \end{cases}$$

It follows that $\hat{h}$ and $\hat{g}$ are $t-1$-step-ahead plans such that $\hat{h} \geq_{SMT} \hat{g}$, $\hat{h} \sim h$ and $\hat{g} \sim g$.[18]

Step 3. If $h, g \in \mathbf{H}_t$, then $h \succsim g$. By Step 2, there exist $\hat{h}, \hat{g} \in \mathbf{H}_{t-1}$ such that $h \sim \hat{h}, g \sim \hat{g}$, and $\hat{h} \geq_{STM} \hat{g}$. Re-applying the same step to $\hat{h}, \hat{g}$ and forward we have that there exist $\overline{h}, \overline{g} \in \mathbf{H}_1$ such that $h \sim \overline{h}, g \sim \overline{g}$ and $\overline{h} \geq_{STM} \overline{g}$. Thus, by Step 1, we have that $h \succsim g$.

---

[16]Here $(h)^{s^{t-1}, t-1}$ denotes the plan $(h_{t-1}(s^{t-1}), h_t(s^{t-1}, \cdot), \ldots)$.

[17]Here, we adopt some abuse of notation. We should write each $\hat{h}_\tau, h_\tau$. etc. as a mapping from Ω; however notice that because of our measurability requirement this notation is meaningful.

[18]The proof of the previous equivalences relies on the re-iteration of the representation till time t and the application of stationarity. These details are available upon request.

Step 4. We have $h \succsim g$. Fix $x \in X$ and define h^t, g^t as follows

$$h^t_\tau = \begin{cases} h_\tau & \tau \le t, \\ x & \tau > t, \end{cases} \text{ and } g^t_\tau = \begin{cases} g_t & \tau \le t, \\ x & \tau > t, \end{cases}$$

for all $t, \tau \ge 0$. We have that $(h^t(\omega))_{t\ge 0}$ and $(g^t(\omega))_{t\ge 0}$ converge to $h(\omega)$ and $g(\omega)$ for all $\omega \in \Omega$, respectively. Since all h^t, g^t belong to $\mathbf{H}_t$ and $h^t \ge_{STM} g^t$, by Step 3, we have that $h^t \succsim g^t$ for all $t \ge 1$. Therefore by continuity of $\succsim$ we obtain $h \succsim g$ as desired.

$[(i) \Rightarrow (ii)]$. First observe that by state-time monotonicity and the one-step-ahead equivalence axioms we have that for all $h, g \in \mathbf{H}$ such that $h_0 = g_0$

$$\left[\forall s \in S,\ h^s \succsim g^s\right] \implies h \succsim g. \tag{11}$$

Indeed, suppose that $h, g \in \mathbf{H}$ are such that $h_0 = g_0$ and $h^s \succsim g^s$ for all $s \in S$. By one-step-ahead equivalence, there exist one-step-ahead acts h^{+1} and g^{+1} such that, $h_0 = h_0^{+1} = g_0^{+1} = g_0$, $h \sim h^{+1}$, $g \sim g^{+1}$, $\left(h^{+1}\right)^s \sim h^s$, and $\left(g^{+1}\right)^s \sim g^s$ for all $s \in S$. Fix $s \in S$, $t, t' \ge 0$, and $\omega, \omega' \in \{s\} \times S^\infty$. Notice that since h^{+1} is a one-step-ahead act, we must have

$$h_t^{+1}(\omega) = h_{t'}^{+1}(\omega')$$

and hence, given the arbitrary nature of t, t' and ω, ω' we have

$$\left(h^{+1}\right)^s = \left(h_0, h_t^{+1}(\omega), h_t^{+1}(\omega), \ldots\right)$$

for all $t \ge 1$, $s \in S$, and $\omega \in \{s\} \times S^\infty$. The same reasoning applies to g^{+1}. Moreover, by transitivity we have

$$\left(h^{+1}\right)^s \sim h^s \succsim g^s \sim \left(g^{+1}\right)^s$$

for all $s \in S$. Thus we have that

$$\left(h_0, h_t^{+1}(\omega), h_t^{+1}(\omega), \ldots\right) \succsim \left(h_0, g_t^{+1}(\omega), g_t^{+1}(\omega), \ldots\right)$$

for all $\omega \in \Omega$ and $t \ge 0$. By stationarity, it follows that

$$\left(h_t^{+1}(\omega), h_t^{+1}(\omega), \ldots\right) \succsim \left(g_t^{+1}(\omega), g_t^{+1}(\omega), \ldots\right)$$

for all $\omega \in \Omega$ and $t \ge 0$. Then, by state-time monotonicity and transitivity we have that

$$h \sim h^{+1} \succsim g^{+1} \sim g.$$

Further, by Lemma 3 there exists an affine, continuous, and non-constant function $u: X \to \mathbb{R}$ such that

$$U : d \mapsto \sum_{t \geq 0} \beta^t u(d_t),$$

represents $\succsim$ on $\mathbf{D}$ with $\beta \in (0,1)$. Fix $h \in \mathbf{H}$. Since C is compact we have that $X = \triangle^\sigma(C)$ is compact as well. Therefore, given that $\succsim$ is a continuous, complete and transitive, we have that there must exist x^*, x_* such that $x^* \succsim x \succsim x_*$ for all $x \in X$. By state-time monotonicity we have that

$$x^* \succsim h \succsim x_*$$

for all $h \in \mathbf{H}$. This implies that the sets

$$\{x \in X : (h_0, x, \dots, x, \dots) \succsim h\} \text{ and } \{x \in X : h \succsim (h_0, x, \dots, x, \dots)\},$$

are not empty. Furthermore, by the continuity of $\succsim$ they are both closed. Since $\succsim$ is a weak order, it holds

$$\{x \in X : (h_0, x, \dots, x, \dots) \succsim h\} \cup \{x \in X : h \succsim (h_0, x, \dots, x, \dots)\} = X.$$

Therefore, since X is connected,[19] we must have that

$$\{x \in X : (h_0, x, \dots, x, \dots) \succsim h\} \cap \{x \in X : h \succsim (h_0, x, \dots, x, \dots)\} \neq \varnothing.$$

which implies that there exists $x_h \in X$ such that $(h_0, x_h, \dots, x_h, \dots) \sim h$. Thus we can define the map

$$V(h) = u(h_0) + \beta U((x_h, x_h, \dots))$$

for all $h \in \mathbf{H}$ and V represents $\succsim$. Now define $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$ as $I_{+1}(V \circ h^1) = U((x_h, x_h, \dots))$. Verifying that I_{+1} is a well-defined function follows the same exact steps of proving its monotonicity (using equalities and equivalences) and hence we prove only the latter.[20] It is easy to see that I_{+1} must be normalized. Now we prove that I_{+1} is also monotone. Suppose that $\xi \geq \hat{\xi}$ for some $\xi, \hat{\xi} \in B(U(\mathbf{D}), S, \Sigma)$. Given that I_{+1} is independent

[19]This follows observing that $X = \triangle^\sigma(C)$ is a convex set endowed with the topology of the weak convergence, and hence it is path-connected.

[20]Notice in particular, that the following holds true

$$B(U(\mathbf{D}), S, \Sigma) = \left\{V(h^1) : h \in H\right\}.$$

of the first period 0, we can assume without loss of generality that there exist $h, \hat{h} \in \mathbf{H}$ such that $h_0 = \hat{h}_0$ and $\xi = V(h^1)$, $\hat{\xi} = V(\hat{h}^1)$. Since V represents $\succsim$, we have that $h^{s,1} \succsim \hat{h}^{s,1}$ for all $s \in S$. Then by stationarity we have

$$h^s = (h_0, h^{s,1}) \succsim (h_0, \hat{h}^{s,1}) = \hat{h}^s$$

for all $s \in S$, and hence, by dynamic consistency (11), we have that $h \succsim \hat{h}$. Therefore we obtain $V(h) \geq V(\hat{h})$ which implies that $U((x_h, x_h, \ldots)) \geq U((x_{\hat{h}}, x_{\hat{h}}, \ldots))$, delivering us the monotonicity of I_{+1}. The continuity of I_{+1} follows from the continuity of $\succsim$. Hence, for all $h \in \mathbf{H}$ we have $V(h) = u(h_0) + \beta I_{+1}(V \circ h^1)$ with I_{+1} being a certainty equivalent and u affine. ■

Before presenting the proof of Theorem 1 we recall an immediate implication of Lemma 5 from Cerreia-Vioglio et al. (2014).

Lemma 4. *Suppose that $K \subseteq \mathbb{R}$ is an interval such that $0 \in \operatorname{int}(K)$. If $I : B(K, \Omega, \mathscr{G})$ satisfies*

$$I(\alpha\xi + (1-\alpha)k) = I(\alpha\xi) + (1-\alpha)k$$

for all $\xi \in B(K, \Omega, \mathscr{G})$, $k \in K$, and $\alpha \in (0,1)$, then it is translation invariant.

Proof of Theorem 1. $[(i) \Rightarrow (ii)]$. By Lemma 3 there exist a discount factor β and an affine, continuous, and non-constant function $u : X \to \mathbb{R}$ such that

$$U : d \mapsto \sum_{t \geq 0} \beta^t u(d_t)$$

represents $\succsim$ on $\mathbf{D}$. This yields that state monotonicity (Axiom I.8) implies state-time monotonicity (Axiom I.9). Indeed, if $h_t(\omega) \succsim g_t(\omega)$ for all $t \geq 0$ and $\omega \in \Omega$, then we have $u(h_t(\omega)) \geq u(g_t(\omega))$ for all $t \geq 0$ and $\omega \in \Omega$. This implies that

$$U(h(\omega)) = \sum_{t \geq 0} \beta^t u(h_t(\omega)) \geq \sum_{t \geq 0} \beta^t u(g_t(\omega)) = U(g(\omega))$$

for all $\omega \in \Omega$. Thus, $h(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, that yields $h \succsim g$ by state monotonicity. Thus, state-time monotonicity holds.

Moreover, by applying the same reasoning at the beginning of the proof of Theorem 3, we obtain that $\succsim$ satisfies one-step-ahead equivalence (Axiom I.10). Therefore by Theorem 3 we have that $\succsim$ admits a recursive representation

$$V : h \mapsto u(h_0) + \beta I_{+1}\left(V \circ h^1\right)$$

for some certainty equivalent $I_{+1} : B(U(\mathbf{D}), S, \Sigma)$. To ease the rest of the exposition we divide the proof in several claims.

Claim 1. There exists a certainty equivalent $I_0 : B(U(\mathbf{D}), \Omega, \mathscr{G}) \to \mathbb{R}$ representing $\succsim$.

Proof of Claim 1. By continuity and state monotonicity, for all $h \in \mathbf{H}$ there exists $d^h \in \mathbf{D}$ such that $h \sim d^h$. This allows us to define $I_0 : B(U(\mathbf{D}), \Omega, \mathscr{G}) \to \mathbb{R}$ as

$$I_0 (U(h)) = U(d^h)$$

for all $h \in \mathbf{H}$. Now we prove that I_0 is a well-defined certainty equivalent. By state monotonicity I_0 is well-defined and monotone. In addition, I_0 is normalized. Indeed, if $k \in U(\mathbf{D})$ we have that there exists $d \in \mathbf{D}$ such that $U(d) = k$ and hence $I_0(k) = I_0(U(d)) = U(d) = k$. Clearly I_0 represents $\succsim$. In conclusion, notice that by continuity of $\succsim$ it follows that I_0 must be continuous. Indeed by the continuity of $\succsim$ there exists a strictly increasing function $f : \mathrm{Im}(I_0 \circ U) \to \mathbb{R}$ such that $f \circ I_0 \circ U$ is continuous and represents $\succsim$. Since u is cardinally unique we have that f must be affine with $f(r) = ar + b$ for some $a > 0$, $b \in \mathbb{R}$, and all $r \in \mathrm{Im}(I_0 \circ U)$. This yields that I_0 must be continuous. Thus I_0 is a certainty equivalent representing $\succsim$. □

Claim 2. The certainty equivalent I_0 is translation invariant.

Before passing to the proof we introduce some notation and make some observations. For all $t \geq 1$, let $\pi_t : \Omega \to \Omega$ be the t-shift forward operator defined as

$$\pi_t(s_1, s_2, \ldots, s_k, \ldots) = (s_{t+1}, s_{t+2}, \ldots, s_{t+k}, \ldots)$$

for all $(s_1, s_2, \ldots, s_k, \ldots) \in \Omega$. Since u is continuous, cardinally unique, and X is compact we can assume that $u(X) = \left[\beta - 1, 1 - \beta\right]$, so that $U(\mathbf{D}) = [-1, 1]$. We denote by $x_*, x^*, x_\circ$ the elements of X such that $u(x_*) = \beta - 1$, $u(x^*) = 1 - \beta$, and $u(x_\circ) = 0$, respectively. To ease notation we denote by x^t the t-vector $(x, \ldots, x)$ for all $t \geq 0$ and $x \in X$.

Proof of Claim 2. The proof consists in defining *auxiliary* certainty equivalents such that I_0 results in nesting them subsequently. These auxiliary functionals will be used to prove the translation invariance of I_0.

Step 1. for all $t \geq 1$ let

$$B_t := \left\{\xi \in B(U(\mathbf{D}), \Omega, \mathscr{G}) : \xi = U(\mathbf{x}, h), \mathbf{x} \in X^{t-1}, h \in \mathbf{H}_1\right\},$$

and observe that by state monotonicity and continuity we can define certainty equivalents[21] $I_t : B_t \to \mathbb{R}$ as $I_t(\xi) = U\left(d^{(\mathbf{x},h)}\right)$ for all $\xi = U(\mathbf{x}, h) \in B_t$, where $d^{(\mathbf{x},h)} \in \mathbf{D}$ satisfies $d^{(\mathbf{x},h)} \sim (\mathbf{x}, h)$.

Step 2. Now observe that for all $t \geq 1$ and $\xi \in B_{t+1}$ such that $\xi = U(x_\circ^t, h)$ stationarity implies that $d^{(x_\circ^t,h)} \sim (x_\circ^t, d^h)$, where $h \sim d^h$. Therefore, we obtain that for all $t \geq 0$,

$$I_{t+1}(\xi) = U\left(x_\circ^t, d^h\right) = \beta^t U\left(d^h\right) = \beta^t I_1\left(\frac{\xi \circ \pi_t^{-1}}{\beta^t}\right). \tag{12}$$

The last equality follows observing that by definition of concatenation we have

$$\begin{aligned}\frac{\xi(\pi_t^{-1}(s_1, s_2, \ldots))}{\beta^t} &= \frac{U(x_\circ^t, h(\pi_t^{-1}(s_{t+1}, s_{t+2}, \ldots)))}{\beta^t} = \sum_{\tau=0} \beta^\tau u\left(h_\tau\left(\pi_t^{-1}(s_{t+1}, s_{t+2}, \ldots)\right)\right) \\ &= \sum_{\tau=0} \beta^\tau u(h_\tau(s_1, s_2, \ldots)) = U(h(s_1, s_2, \ldots))\end{aligned}$$

for all $(s_1, s_2, \ldots) \in \Omega$.

Step 3. Fix $t \geq 1$ and $\xi \in B_{t+1}$, $k \in U(\mathbf{D})$ such that $k + \xi \in B_{t+1}$, $k = \sum_{\tau=0}^{t-1} \beta^\tau u(x_\tau)$, and $\xi = \sum_{\tau \geq t} \beta^\tau u(h_\tau)$ for some $h \in \mathbf{H}_1$ and $\mathbf{x} \in X^t$. By stationarity $(\mathbf{x}, h) \sim (\mathbf{x}, d^h)$ where $d^h \in \mathbf{D}$ is such that $h \sim d^h$, so by Step 2 we obtain

$$I_{t+1}(k + \xi) = \sum_{\tau=0}^{t-1} \beta^\tau u(x_\tau) + \beta^t I_1\left(\frac{\xi \circ \pi_t^{-1}}{\beta^t}\right) = k + I_{t+1}(\xi). \tag{13}$$

Condition (13) implies that for all $\xi = U(h) \in B_1$ we have

$$\begin{aligned}I_1\left(u(h_0) + \sum_{\tau \geq 1} \beta^\tau u(h_\tau)\right) &= \frac{1}{\beta} I_2\left(\beta u(h_0) + \sum_{\tau \geq 1} \beta^{\tau+1} u(h_\tau \circ \pi_1)\right) \\ = u(h_0) + \frac{1}{\beta} I_2\left(\sum_{\tau \geq 1} \beta^{\tau+1} u(h_\tau \circ \pi_1)\right) &= u(h_0) + I_1\left(\sum_{\tau \geq 1} \beta^\tau u(h_\tau)\right).\end{aligned}$$

Step 4. Fix $t \geq 1$ and $\xi = \sum_{\tau \geq 0} \beta^\tau u(h_\tau)$ for some $h \in \mathbf{H}_t$. We have that $I_0(\xi) = I_1(\xi)$ when $t = 1$. For $t \geq 2$, we have that

$$I_0(\xi) = I_1\left(I_2\left(\cdots\left(I_t\left(\xi\right)\right)\right)\right). \tag{14}$$

To see that, notice that by definition $I_0(\xi) = V(h)$ and for all $s_1, s_2, \ldots, s_{t-1} \in S$ we have

$$I_t\left(\xi\left(s_1, s_2, \ldots, s_{t-1}, \cdot\right)\right) = V\left(h\left(s_1, s_2, \ldots, s_{t-1}, \cdot\right)\right).$$

[21] The proof that such functionals are certainty equivalents follows the same steps of the proof of Claim 1 and hence is omitted.

Consequently, for all $s_1, s_2, \ldots, s_{t-2} \in S$, we have that

$$I_{t-1}\left(I_t\left(\xi\left(s_1, s_2, \ldots, s_{t-2}, \cdot, \cdot\right)\right)\right) = V\left(h\left(s_1, s_2, \ldots, s_{t-2}, \cdot, \cdot\right)\right)$$

and proceeding backwards equality (14) follows.

Step 5. Fix arbitrarily $h \in \mathbf{H}$ and define the sequence $(h^t)_{t\geq 0}$ as $h^t = (h_0, \ldots, h_t, x_\circ, x_\circ, \ldots)$ for all $t \geq 0$. Clearly, $h^t \in \mathbf{H}_t$ for all $t \geq 0$. By the previous steps, we obtain

$$\begin{aligned} I_0\left(u(h_0) + \sum_{\tau=1}^{t} \beta^\tau u(h_\tau)\right) &= I_0\left(u(h_0^t) + \sum_{\tau\geq 1} \beta^\tau u(h_\tau^t)\right) \\ &= I_1\left(I_2\left(\cdots\left(I_t\left(u(h_0^t) + \sum_{\tau\geq 1} \beta^\tau u(h_\tau^t)\right)\right)\right)\right) \\ &= u(h_0) + I_0\left(\sum_{\tau=1}^{t} \beta^\tau u(h_\tau)\right) \end{aligned}$$

for all $t \geq 0$. By continuity of I_0, it follows

$$\begin{aligned} I_0\left(u(h_0) + \sum_{\tau\geq 1} \beta^\tau u(h_\tau)\right) &= \lim_{t\to\infty} I_0\left(u(h_0) + \sum_{\tau=1}^{t} \beta^\tau u(h_\tau)\right) \\ &= u(h_0) + I_0\left(\sum_{\tau\geq 1} \beta^\tau u(h_\tau)\right). \end{aligned}$$

Step 6. By the previous step, we obtain that I_0 satisfies: for all $\xi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$, $\alpha \in [0,1]$, and $k \in U(\mathbf{D})$

$$I_0(\alpha\xi + (1-\alpha)k) = I_0(\alpha\xi) + (1-\alpha)k.$$

To see this, assume first that $k \in (-1,1)$ and let $\xi = U(h)$. Since u is affine and continuous, and X is convex, there exists a finite sequence $(d_\tau)_{t=0}^{T}$ in X such that $k = \sum_{\tau=0}^{T} \beta^\tau u(d_\tau)$. Define $g \in \mathbf{H}$ as $g_t = \alpha h_t + (1-\alpha) d_t$ for all $0 \leq t \leq T$ and $\alpha h_t + (1-\alpha) x_\circ$ for all $t > T$. By repeatedly applying the previous steps, the affinity of u, and using the fact that for all $x \in X$ and $t \geq 0$ there exists $y \in X$ such that $\beta^t u(x) = u(y)$[22] we

[22] In greater detail, let $y = \beta^t x + (1-\beta^t) x_\circ$, the affinity of u yields $u(y) = \beta^t u(x)$.

obtain

$$
\begin{aligned}
I_0(\alpha\xi+(1-\alpha)k) &= I_0\left(\sum_{t\geq 0}\beta^t u(g_t)\right) = u(g_0)+I_0\left(\sum_{t\geq 1}\beta^t u(g_t)\right)\\
&= u(g_0)+(1-\alpha)\sum_{\tau=1}^{T}\beta^\tau u(d_\tau)+I_0\left(\alpha\sum_{\tau\geq 1}\beta^\tau u(h_\tau)\right)\\
&= \alpha u(h_0)+(1-\alpha)u(d_0)+(1-\alpha)\sum_{\tau=1}^{T}\beta^\tau u(d_\tau)+I_0\left(\alpha\sum_{\tau\geq 1}\beta^\tau u(h_\tau)\right)\\
&= u(\alpha h_0+(1-\alpha)x_\circ)+(1-\alpha)u(d_0)+(1-\alpha)\sum_{\tau=1}^{T}\beta^\tau u(d_\tau)+I_0\left(\sum_{\tau\geq 1}\beta^\tau u(\alpha h_\tau+(1-\alpha)x_\circ)\right)\\
&= (1-\alpha)\sum_{\tau=0}^{T}\beta^\tau u(d_\tau)+I_0\left(\alpha\sum_{\tau\geq 0}\beta^\tau u(h_\tau)\right) = I_0(\alpha\xi)+(1-\alpha)k.
\end{aligned}
$$

If $k\in\{-1,1\}$, then the statement follows by taking a sequence $(k_n)_{n\geq 0}$ in $U(\mathbf{D})\setminus\{-1,1\}$ such that $\lim k_n = k$ and applying continuity.

Thus, by Lemma 4, I_0 is translation invariant. □

Now we prove that I_0 and I_{+1} must satisfy condition (4).

Claim 3. For all $\xi\in B(U(\mathbf{D}),\Omega,\mathscr{G})$ we have

$$
I_0(\xi) = \beta I_{+1}\left(\frac{1}{\beta}I_0\left(\xi^1\right)\right).
$$

Proof of Claim 3. We argue that it is enough to show the equality for all $\xi\in B(U(\mathbf{D}),\Omega,\mathscr{G})$ with $\xi=\sum_{t\geq 1}\beta^t u(h_t)$ for some $h\in\mathbf{H}_2$. Indeed, suppose for now that this observation is true and fix $\xi\in B(U(\mathbf{D}),\Omega,\mathscr{G})$ with $\xi=\sum_{t\geq 0}\beta^t u(h_t)$ for some $h\in\mathbf{H}$. First notice that it is without loss of generality to assume that $u(h_0)=0$.[23] Since $\succsim$ admits a recursive representation, it satisfies one-step-ahead equivalence. This implies that there exists $\tilde{h}\in\mathbf{H}_1\subseteq\mathbf{H}_2$ such that $\tilde{h}_0=h_0$, $\tilde{h}\sim h$, and $\tilde{h}^s\sim h^s$ for all $s\in S$. By stationarity we have that

[23] To deal with a general element $\xi=\sum_{t\geq 0}\beta^t u(h_t)$ we could work directly with the unique monotone, normalized, and translation invariant extension of I_0 denoted by $\hat{I}_0: B(U(\mathbf{D}),\Omega,\mathscr{G})+\mathbb{R}\to\mathbb{R}$ defined as $\hat{I}_0(\xi+k)=I_0(\xi)+k$ for all $\xi\in B(U(\mathbf{D}),\Omega,\mathscr{G})$ and $k\in\mathbb{R}$. In particular, the proof would change by first proving generalized rectangularity for all $\xi\in B(U(\mathbf{D}),\Omega,\mathscr{G})$ with $\xi=\sum_{t\geq 1}\beta^t u(h_t)$ for some $h\in\mathbf{H}$, then passing to prove the translation invariance of I_{+1} with respect to all such ξ. Once having done that, one can prove generalized rectangularity for $\hat{I}_0$ using the analogously defined extension $\hat{I}_{+1}$ of I_{+1}. In conclusion, one can use the full generalized rectangularity to prove the complete translation invariance of I_{+1}. We omit such additional steps.

$\tilde{h}^{s,1} \sim h^{s,1}$ for all $s \in S$ and hence

$$\begin{aligned} I_0(\xi) = I_0(U(h)) = I_0\left(U\left(\tilde{h}\right)\right) &= \beta I_{+1}\left(\frac{1}{\beta} I_0\left(U\left(\tilde{h}^1\right)\right)\right) \\ &= \beta I_{+1}\left(\frac{1}{\beta} I_0\left(U(h^1)\right)\right) = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right). \end{aligned}$$

The third equality follows from generalized rectangularity for $h \in \mathbf{H}_2$ and the second-to-last from the equivalence $\tilde{h}^{s,1} \sim h^{s,1}$ for all $s \in S$ and the fact that $I_0 \circ U$ represents $\succsim$.

Thus, we prove that generalized rectangularity (4) holds for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$ with $\xi = \sum_{t \geq 1} \beta^t u(h_t)$ for some $h \in \mathbf{H}_2$. Fix such a $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. By definition of I_0 we have that

$$I_0(\xi) = I_0(U(h)) = U(d^h) = V(h) = u(h_0) + \beta I_{+1}\left(V \circ h^1\right)$$

and since $u(h_0) = 0$, we have $I_0(\xi) = \beta I_{+1}(V \circ h^1)$. Now notice that since $h \in \mathbf{H}_2$ we have that $h^{s,1} \in \mathbf{H}_1$ for all $s \in S$. This yields that

$$\begin{aligned} V\left(h^{s,1}\right) &= I_1\left(u(h_1(s,\ldots)) + \beta u(h_2(s,\ldots)) + \ldots\right) = I_1\left(\frac{\xi^{s,1}}{\beta}\right) \\ &= \frac{1}{\beta} I_2\left(\xi^{s,1}\right) = \frac{1}{\beta} I_1\left(I_2\left(\xi^{s,1}\right)\right) = \frac{1}{\beta} I_0\left(\xi^{s,1}\right) \end{aligned}$$

for all $s \in S$, where the second-to-last equality follows from the normalization of I_1 and the last follows from Step 4. This implies that

$$I_0(\xi) = \beta I_{+1}(V \circ h^1) = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right).$$

Thus, we conclude that

$$I_0(\xi) = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right). \tag{15}$$

holds for all $\xi \in B(U(\mathbf{D}), \Omega, \mathcal{G})$. □

Finally, we prove that translation invariance of I_{+1} is implied by generalized rectangularity (15).

Claim 4. The certainty equivalent I_{+1} is translation invariant.

Proof of Claim 4. Suppose by contradiction that $I_{+1}(\xi + k) \neq I_{+1}(\xi) + k$ for some $\xi \in B(U(\mathbf{D}), S, \Sigma)$ and $k \in U(\mathbf{D})$ with $\xi + k \in B(U(\mathbf{D}), S, \Sigma)$. Then it is routine to find $\varphi \in$

$B(U(\mathbf{D}),\Omega,\mathscr{G})$, $w \in U(\mathbf{D})$ such that $\xi = \frac{1}{\beta} I_0\left(\varphi^1\right)$ and $k = w/\beta$.[24] Clearly by translation invariance of I_0, we must have that $\xi + k = \frac{1}{\beta} I_0\left(\varphi^1 + w\right)$. Thanks to these observations, applying generalized rectangularity (15) we have

$$
\begin{aligned}
I_0\left(\varphi + w\right) &= \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\varphi^1 + w\right)\right) = \beta I_{+1}\left(\xi + k\right) \\
&\neq \beta I_{+1}\left(\xi\right) + \beta k = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\varphi^1\right)\right) + \beta k = I_0(\varphi) + w
\end{aligned}
$$

contradicting the translation invariance of I_0. □

[$(ii) \Rightarrow (i)$] Given our Theorem 3 we only need to prove the necessity of dynamic consistency and state monotonicity. Since I_0 and I_{+1} are monotone, the claim follows. ■

We denote by $C_b(\mathbf{H})$ the set of continuous and bounded functions from $\mathbf{H}$ to $\mathbb{R}$. For all $\beta \in (0,1)$ and continuous $u : X \to \mathbb{R}$, the function $U : \mathbf{D} \to \mathbb{R}$ is defined as $U(d) = \sum_{t\geq 0} \beta^t u(d_t)$ for all $d \in \mathbf{D}$. We denote with $C_b(\mathbf{H}, U(\mathbf{D}))$ the set of continuous and bounded functions taking values in $U(\mathbf{D})$. By $B_{ce}(U(\mathbf{D}),\Omega,\mathscr{G})$ we denote the set of translation invariant certainty equivalents with domain $B(U(\mathbf{D}),\Omega,\mathscr{G})$. The proof of Theorem 2 relies on the following preliminary result.

Lemma 5. *Let $u : X \to \mathbb{R}$ be an affine, continuous, and non-constant function and $\beta \in (0,1)$. Suppose that $I_{+1} : B(U(\mathbf{D}), S, \Sigma) \to \mathbb{R}$ is a translation invariant certainty equivalent. The following are equivalent*

(i) there exists $V \in C_b(\mathbf{H}, U(\mathbf{D}))$, such that

$$
V(h) = u(h_0) + \beta I_{+1}\left(V \circ h^1\right) \text{ for all } h \in \mathbf{H}, \tag{16}
$$

(ii) there exists $I_0 \in B_{ce}(U(\mathbf{D}),\Omega,\mathscr{G})$ such that

$$
I_0(\xi) = \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right) \text{ for all } \xi \in B(U(\mathbf{D}),\Omega,\mathscr{G}). \tag{17}
$$

Proof. Given V satisfying (16), define $\succsim$ on $\mathbf{H}$ as follows: $h \succsim g$ if and only if $V(h) \geq V(g)$. Then $\succsim$ satisfies axioms I.1, I.9, I.2-I.10. Because I_{+1} is translation invariant, by applying the same reasoning as Proposition 4 in Bommier et al. (2017) it follows that $\succsim$ satisfies state monotonicity. Hence by Theorem 1, we obtain that there exists $I_0 \in B_{ce}(U(\mathbf{D}),\Omega,\mathscr{G})$ satisfying (17).

[24] For instance define $\varphi^{s,1}(\omega) = \beta\xi(s)$ for all $\omega \in \Omega$ and $s \in S$. Then, use the normalization of I_0.

Conversely, given $I_0 \in B_{ce}(U(\mathbf{D}), \Omega, \mathscr{G})$ satisfying (17), define $V \in C_b(\mathbf{H}, U(\mathbf{D}))$ as

$$V(h) = I_0(U(h)),$$

for all $h \in \mathbf{H}$. Observe that for all $h \in \mathbf{H}$ we can use (17) and translation invariance to find $h^{+1} \in \mathbf{H}_1$ such that $h_0 = h_0^{+1}$, $V(h) = V\left(h^{+1}\right)$ and $V\left(h^{s,1}\right) = V\left(\left(h^{+1}\right)^{s,1}\right)$ for all $s \in S$.[25] Hence, using (17) and the fact that I_0 is normalized and translation invariant we obtain

$$\begin{aligned} V(h) = V(h^{+1}) = I_0(U(h^{+1})) &= u(h_0) + I_0\left(\beta u(h_1^{+1}) + \beta^2 u(h_2^{+1}) + \ldots\right) \\ &= u(h_0) + \beta I_{+1}\left(u(h_1^{+1}) + \beta u(h_2^{+1}) + \ldots\right) = u(h_0) + \beta I_{+1}(V \circ h^1), \end{aligned}$$

so that $V \in C_b(\mathbf{H}, U(\mathbf{D}))$ satisfies (16) as desired.[26] ∎

Proof of Theorem 2. We endow $C_b(\mathbf{H})$ with the metric d_∞ defined by $d_\infty(V, V') = \sup_{h \in \mathbf{H}} |V(h) - V'(h)|$ for all $V, V' \in C_b(\mathbf{H})$. Observe that $(C_b(\mathbf{H}), d_\infty)$ is complete and $C_b(\mathbf{H}, U(\mathbf{D})) \subseteq C_b(\mathbf{H})$ is closed. Now we define the operator $T : C_b(\mathbf{H}, U(\mathbf{D})) \to C_b(\mathbf{H}, U(\mathbf{D}))$ as

$$T(V)(h) = u(h_0) + \beta I_{+1}\left(V \circ h^1\right) \text{ for all } h \in \mathbf{H} \text{ and } V \in C_b(\mathbf{H}, U(\mathbf{D})).$$

Observe that T is a well-defined selfmap since u, V, I_{+1} are continuous. Moreover, it satisfies $T(V) \geq T(V')$ whenever $V, V' \in C_b(\mathbf{H}, U(\mathbf{D}))$ satisfy $V \geq V'$. Further, because I_{+1} is translation invariant, we have

$$T(V + k) = T(V) + \beta k$$

for all $V \in C_b(\mathbf{H}, U(\mathbf{D}))$ and $k \in \mathbb{R}$ such that $V + k \in C_b(\mathbf{H}, U(\mathbf{D}))$. Reproducing step-by-step the proof of Blackwell's contraction theorem (Blackwell, 1965), it follows that T is a

[25] To see this, take $h \in \mathbf{H}_2$. Then, by (17). the translation invariance and normalization of I_0,

$$\begin{aligned} V\left(h^{s,1}\right) &= u\left(h_1(s, \ldots)\right) + I_0\left(\beta u\left(h_2((s, \ldots)\right) + \beta^2 u(h_3(s, \ldots)) + \ldots\right) \\ &= u\left(h_1(s, \ldots)\right) + \beta I_{+1}\left(u(h_2(s, \ldots)) + \beta u(h_3(s, \ldots)) + \ldots\right) = u\left(h_1(s, \ldots)\right) + \beta U((x^s, x^s, \ldots)), \end{aligned}$$

for some $x^s \in X$ and all $s \in S$. The claim follows by letting $h_0^{+1} = h_0$, $h_t^{+1}(s, \ldots) = x^s$ for all $t \geq 2$ and $s \in S$. The claim can be extended to arbitrary $h \in \mathbf{H}_t$ for all $t \geq 2$ and by continuity to every $h \in \mathbf{H}$.

[26] Notice that the second-to-last and last equalities follow from the normalization of I_0, the fact that $h^{+1} \in \mathbf{H}_1$, and

$$\frac{1}{\beta} I_0\left(\beta u(h_1^{+1}(s, \ldots)) + \beta^2 u(h_2^{+1}(s, \ldots)) + \ldots\right) = u(h_1^{+1}(s, \ldots)) + \beta u(h_2^{+1}(s, \ldots)) + \ldots = V\left(\left(h^{+1}\right)^{s,1}\right)$$

for all $s \in S$.

β-contraction with respect to d_∞.[27] Therefore, given that $C_b(\mathbf{H}, U(\mathbf{D}))$ is a closed subset of a complete metric space, there exists a unique $V^* \in C_b(\mathbf{H}, U(\mathbf{D}))$ satisfying (16). Hence, by Lemma 5, it follows that there exists a unique I_0^* satisfying (17), as desired. ■

Numerical procedure. Observe that I_0^* in the proof of Theorem 2 can be obtained implementing the following algorithm. For all $I_0^0 \in B_{ce}(U(\mathbf{D}), \Omega, \mathscr{G})$, define $V^0 \in C_b(\mathbf{H}, U(\mathbf{D}))$ by $V^0(h) = I_0^0(U(h))$ for all $h \in \mathbf{H}$. The sequence defined by $T^0 = V^0$ and $T^n = T(T^{n-1})$ for all $n \geq 1$ converges to $V^* \in C_b(\mathbf{H}, U(\mathbf{D}))$ satisfying (16). Therefore by Lemma 5 we construct I_0^* satisfying (17) by setting $I_0^*(\xi) = V^*(h)$ for all $\xi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$ with $\xi = U(h)$.

Proof of Corollary 3. Suppose that I_0 and I_{+1} satisfy (4). Then, since both I_0 and I_{+1} are positively homogeneous, it follows that generalized rectangularity is equivalent to

$$\begin{aligned}\min_{P\in\mathscr{P}} \mathbb{E}_P[\xi] = I_0(\xi) = I_{+1}\left(I_0(\xi^1)\right)\\ = \min_{\ell\in\mathscr{L}} \mathbb{E}_\ell\left[\min_{P\in\mathscr{P}} \mathbb{E}_P\left[\xi^1\right]\right] = \min_{\ell\in\mathscr{L}} \sum_{s\in S} \ell(s) \min_{P^s\in\mathscr{P}} \mathbb{E}_{P^s}\left[\xi^1\right]\end{aligned}$$

for all $\xi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$. This last condition is equivalent to the $\mathscr{L}$-rectangularity of $\mathscr{P}$.[28] ■

Proof of Corollary 2. By Theorem 1 there exist I_0 and I_{+1} that satisfy translation invariance. Observe that by (6) I_0 is quasi-concave, and so by Lemma 25 in Maccheroni et al. (2006a) it is also concave. Now we prove that also I_{+1} must be quasi-concave, and hence concave thanks to its translation invariance. Suppose by contradiction that I_{+1} is not quasi-concave. Then, there exist $\xi, \xi' \in B(U(\mathbf{D}), S, \Sigma)$ and $\alpha \in (0,1)$ such that $I_{+1}(\xi) = I_{+1}(\xi')$ and

$$I_{+1}\left(\alpha\xi + (1-\alpha)\xi'\right) < I_{+1}(\xi). \tag{18}$$

Let $\varphi, \psi \in B(U(\mathbf{D}), \Omega, \mathscr{G})$ be such that $\varphi^{s,1}(\omega) = \beta\xi(s)$ and $\psi^{s,1}(\omega) = \beta\xi'(s)$ for all $s \in S$ and $\omega \in \Omega$. Then, since I_0 is normalized we have that

$$I_0\left(\alpha\varphi^{s,1} + (1-\alpha)\psi^{s,1}\right) = \beta\left[\alpha\xi(s) + (1-\alpha)\xi'(s)\right],\ I_0\left(\varphi^{s,1}\right) = \beta\xi(s),\ I_0\left(\psi^{s,1}\right) = \beta\xi'(s) \tag{19}$$

[27] To be more precise, all these steps should be applied to the unique, monotone operator $\bar{T}: C_b(\mathbf{H}, U(\mathbf{D})) + \mathbb{R} \to C_b(\mathbf{H}, U(\mathbf{D})) + \mathbb{R}$ that extends T and satisfies $\bar{T}(V+k) = \bar{T}(V) + \beta k$ for all $V \in C_b(\mathbf{H}, U(\mathbf{D})) + \mathbb{R}$ and $k \in \mathbb{R}$. Clearly, $\bar{T}$ is defined as $\bar{T}(V+k) = T(V) + \beta k$ for all $V \in C_b(\mathbf{H}, U(\mathbf{D}))$ and $k \in \mathbb{R}$. Reproducing step-by-step the proof of Blackwell's contraction theorem we get that $\bar{T}$ is a β-contraction and so must be T.

[28] For a proof the reader can consult Amarante and Siniscalchi (2019). We can provide it upon request.

for all $s \in S$. Then, by generalized rectangularity, (18), and (19) we have that

$$\begin{aligned}I_0(\alpha\varphi + (1-\alpha)\varphi') &= \beta I_{+1}\left(\frac{1}{\beta} I_0\left(\alpha\varphi^1 + (1-\alpha)\psi^1\right)\right) = \beta I_{+1}\left(\alpha\xi + (1-\alpha)\xi'\right) \\ &< \beta I_{+1}(\xi) = \beta\left[\alpha I_{+1}(\xi) + (1-\alpha) I_{+1}(\xi')\right] \\ &= \alpha\beta I_{+1}\left(\frac{1}{\beta} I_0\left(\varphi^1\right)\right) + (1-\alpha)\beta I_{+1}\left(\frac{1}{\beta} I_0\left(\psi^1\right)\right) = \alpha I_0(\varphi) + (1-\alpha) I_0(\psi)\end{aligned}$$

which contradicts the concavity of I_0. Thus, I_{+1} must be concave. By Theorem 3 in Maccheroni et al. (2006a) we obtain the desired variational representations. In particular, there exist cost functions $c_0 : \triangle(\Omega) \to [0,\infty]$ and $c_{+1} : \triangle(S) \to [0,\infty]$ such that

$$I_0 = \min_{P\in\triangle(\Omega)} \{\mathbb{E}_P[\cdot] + c_0(P)\} \text{ and } I_{+1} = \min_{\ell\in\triangle(S)} \{\mathbb{E}_\ell[\cdot] + c_{+1}(\ell)\}.$$

To prove that the no-gain condition implies generalized rectangularity (4), fix a non-negative $\xi \in B_0(U(\mathbf{D}),\Omega,\mathscr{G})$ with $\xi = \sum_{i=1}^n \xi_i 1_{A_i}$ for some $\mathscr{G}$-measurable partition $(A_i)_{i=1}^n$. Then, we have that

$$\begin{aligned}\beta I_{+1}\left(\frac{1}{\beta} I_0\left(\xi^1\right)\right) &= \beta \min_{\ell\in\triangle(S)} \left\{\sum_{s\in S} \ell(s) \min_{P_s\in\triangle(\Omega)} \left\{\sum_{i=1}^n P_s(A_{i,s})\frac{\xi_i}{\beta} + \frac{1}{\beta} c_0(P_s)\right\} + c_{+1}(\ell)\right\} \\ &= \min_{\ell\in\triangle(S)} \left\{\sum_{s\in S} \ell(s) \min_{P_s\in\triangle(\Omega)} \left\{\sum_{i=1}^n P_s(A_{i,s})\xi_i + c_0(P_s)\right\} + \beta c_{+1}(\ell)\right\} \\ &= \min_{\ell\in\triangle(S)} \min_{(P_s)_{s\in S}\in\triangle(\Omega)^S} \left\{\sum_{s\in S}\sum_{i=1}^n \ell(s) P_s(A_{i,s})\xi_i + \sum_{s\in S} \ell(s) c_0(P_s) + \beta c_{+1}(\ell)\right\} \\ &= \min_{P\in\triangle(\Omega)} \left\{\sum_{s\in S}\sum_{i=1}^n P_{+1}(s) P_s(A_{i,s})\xi_i + \sum_{s\in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1})\right\} \\ &= \min_{P\in\triangle(\Omega)} \left\{\sum_{i=1}^n P(A_i)\xi_i + c_0(P)\right\} \\ &= I_0(\xi).\end{aligned}$$

where the secont-to-last equality is implied by (7). Generalized rectangularity (4) for all $\xi \in B(U(\mathbf{D}),\Omega,\mathscr{G})$ then holds by continuity and monotone convergence theorem. ■

The following proposition highlights a further connection between our generalized rectangularity and the no-gain condition.

Proposition 3. *Suppose that for all $P \in \triangle(\Omega)$,*

$$c_0(P) = \sup_{h\in\mathbf{H}} \left\{U(d^h) - \mathbb{E}_P\left[\sum_{t\geq 0} \beta^t u(h_t)\right]\right\}.$$

Then generalized rectangularity (4) implies the inequality

$$c_0(P) \leq \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1})$$

for all $P \in \triangle(\Omega)$. Conversely, (7) implies generalized rectangularity (4).

Proof. By generalized rectangularity, we have that

$$\min_{P \in \triangle(\Omega)} \left\{ \sum_{s \in S} \sum_{i=1}^{n} P_{+1}(s) P_s(A_{i,s}) \xi_i + \sum_{s \in S} P_{+1}(s) c_{+1}(P_s) + \beta c_{+1}(\ell) \right\}$$
$$= \min_{P \in \triangle(\Omega)} \left\{ \sum_{i=1}^{n} P(A_i) \xi_i + c_0(P) \right\},$$

for all $\xi \in B_0(U(\mathbf{D}), \Omega, \mathscr{G})$. By Theorem 3 in Maccheroni et al. (2006a) we have that

$$c_0(P) \leq \sum_{s \in S} P_{+1}(s) c_0(P_s) + \beta c_{+1}(P_{+1}).$$

The other side of the claim was already shown in the proof of Corollary 2. ■

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