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Foundations of ambiguity models under symmetry: α-MEU and smooth ambiguity ☆

Peter Klibanoff [a,*], Sujoy Mukerji [b], Kyoungwon Seo [c], Lorenzo Stanca [a]

[a] *Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, Evanston, IL, USA*

[b] *School of Economics and Finance, Queen Mary University of London, London E1 4NS, UK*

[c] *Business School, Seoul National University, Seoul, South Korea*

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Abstract

The α-MEU model and the smooth ambiguity model are two popular models in decision making under ambiguity. However, the axiomatic foundations of these two models are not completely understood. We provide axiomatic foundations of these models in a symmetric setting with a product state space S^{∞}. This setting allows marginals over S to be linked behaviorally with (limiting frequency) events. Bets on such events are shown to reveal the i.i.d. measures that are relevant for the decision maker's preferences and appear in the representations. By characterizing both models within a common framework, it becomes possible to better compare and relate them.

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* Corresponding author.

E-mail addresses: peterk@kellogg.northwestern.edu (P. Klibanoff), s.mukerji@qmul.ac.uk (S. Mukerji), seo8240@snu.ac.kr (K. Seo), lorenzo.stanca@kellogg.northwestern.edu (L. Stanca).

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1. Introduction

In decision making under ambiguity, an important concern is modeling and discriminating between "perception" of ambiguity and ambiguity attitudes. Two popular models that have been described as allowing for a distinction between the two are the α-MEU model and the smooth ambiguity model. The α-MEU model ranks acts f according to the criterion

$$V(f) = \alpha \min_{p \in C} \int u(f)\, dp + (1-\alpha) \max_{p \in C} \int u(f)\, dp, \tag{1}$$

while the smooth ambiguity model ranks acts f according to

$$U(f) = \int \phi\left(\int u(f) dp \right) \mu(p). \tag{2}$$

In the former, ambiguity perception is captured by the set C and attitudes are described by the parameter α. For the latter, the second-order measure μ captures ambiguity perception while the curvature of ϕ describes the ambiguity attitude. Part of their popularity is explained by the ability of separating between the two. However, the axiomatic foundations of these two models are not yet completely well-understood. Importantly, the fact that there is no axiomatization of the two models in a common framework has inhibited comparison of the two.[1] Existing axiomatizations of the α-MEU model (e.g., Ghirardato et al., 2004; Kopylov, 2003; Gul and Pesendorfer, 2015), as the discussion in Section 1.1 describes, characterize different special cases of the model. The axiomatization of the smooth ambiguity model in Klibanoff et al. (2005) has been criticized for using second order acts (e.g., see the comment by Epstein, 2010 and the reply by Klibanoff et al., 2012). Seo's (2009) related axiomatization did not use second order acts, but his result is not able to uniquely separate the function ϕ from the prior μ.

In this paper, we axiomatize these two models in a common framework under a symmetry assumption on preferences. In particular, when the state space Ω has the product structure $\Omega = S^\infty$, we axiomatize a version of the α-MEU model that takes the form

$$V(f) \equiv \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp + (1-\alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp, \tag{3}$$

where D is a finite set of probability measures over S, $\alpha \in [0, 1]$, and u is a non-constant, affine utility function. In words, in this special case of the α-MEU model the set C only contains i.i.d. probability measures having marginal distributions contained in the set D. The smooth ambiguity model we axiomatize under symmetry takes the form

$$U(f) = \int \phi\left(\int u(f) d\ell^\infty \right) \mu(\ell), \tag{4}$$

where μ is a probability measure over $\Delta(S)$, u is a non-constant, affine utility function and ϕ is continuous and strictly increasing. Furthermore, if the support of μ is not finite then ϕ must satisfy a Lipschitz-type condition.

The symmetric setting we consider is natural for many empirically grounded applications of ambiguity. Indeed, many economic models impose constraints on the agents' preferences so that

[1] Epstein (2010), discussing critically the smooth ambiguity model, remarked that "However, because of its problematic foundations, the behavioral content of the model and how it differs from multiple priors, for example, are not clear."

they reflect some type of calibration of perceived ambiguity to external data (see e.g., Hansen and Sargent, 2008 for motivation and discussion). For instance, in an asset pricing model, the modeler may want to impose the restriction that an investor seeks to make her portfolio robust against only a limited set of stochastic processes that pass certain tests of inference on past data. Typically, such tests rest on the assumption that past and current data generating processes are (at least, conditionally) exchangeable, thus invoking symmetry.

To illustrate the implications of our results, we consider three thought experiments:

Consider an individual with preference $\succ$ who can bet on two sources of uncertainty. The first is an urn with 100 balls divided in an unknown way between black balls and white balls. The other source of uncertainty is the return on TESLA stock. More precisely, the individual can bet on the results of repeated draws from the urn and repeated daily returns of the stock. The state space is $S = (\{B, W\} \times V)^\infty$, where V is an interval containing the possible daily returns. The individual has to choose among the following bets:

(i) bets h and l, where h pays \$100 if 50% to 60% of the daily returns are between 0% and 1% and \$0 otherwise, while l pays \$90 if 20% to 30% of the daily returns are between 0% and 1% and \$0 otherwise;
(ii) bets H and L. Here, H pays \$100 if 50% to 60% of the daily returns are between 0% and 1% and, if neither 50% to 60% nor 20% to 30% of the daily returns are between 0% and 1%, pays \$90 if 30% to 40% of the balls drawn from the urn are black, and \$0 otherwise. L pays \$90 if 20% to 30% of the daily returns are between 0% and 1% and, if neither 50% to 60% nor 20% to 30% of the daily returns are between 0% and 1%, pays \$90 if 30% to 40% of the balls drawn from the urn are black, and \$0 otherwise.

Note that in such a setting the assumption of symmetry is realistic, since the change in a stock price is typically modeled as i.i.d.[2]

Let E^h and E^l denote the events that 50% to 60% and 20% to 30% of the daily returns are between 0% and 1%, respectively, and let B be the event that 30% to 40% of the balls drawn from the urn are black. If an individual follows the smooth ambiguity model, $h \succ l$ implies that $H \succ L$. This is a consequence of the smooth ambiguity model as in (4) necessarily satisfying the sure-thing principle when restricted to bets over long-run frequencies. To see that this requirement forces $H \succ L$ when $h \succ l$, notice that H and L are constructed from h and l by changing the common payoff on the event $B \cap (E^l \cup E^h)^c$ from \$0 to \$90. On the other hand, if an individual uses the α-MEU model as in (3), then the pattern of preference $h \succ l$ and $L \succ H$ is allowed. To illustrate, let u be the identity, $\alpha = \frac{3}{4}$, and the set of measures, C, be $\{\ell_1^\infty, \ell_2^\infty\}$ for some $\ell_1, \ell_2 \in \Delta(\{\{B, W\} \times V\})$ with $\ell_1^\infty(E^h) = \frac{1}{2} = \ell_2^\infty(E^l)$, $\ell_1^\infty(B) = 1$ and $\ell_2^\infty(E^h) = \ell_2^\infty(B) = 0 = \ell_1^\infty(E^l)$. These α-MEU preferences imply that $h \succ l$ and $L \succ H$, a violation of the sure-thing principle when restricted to bets over long-run frequencies. To see that $h \succ l$, observe that $\max_{p \in C} p(E^h) = \max_{p \in C} p(E^l) = \frac{1}{2}$ and $\min_{p \in C} p(E^h) = \min_{p \in C} p(E^l) = 0$, while $\frac{1}{2}100 = 50 > 45 = \frac{1}{2}90$. However, $L \succ H$ due to the fact that L, which pays \$90 if E^l or $B \cap (E^l \cup E^h)^c$ occurs, provides a hedge against ambiguity, while H, which pays \$100 if E^h or \$90 if $B \cap (E^l \cup E^h)^c$ occurs, does not. Indeed, according to all measures in C, the event $(E^l \cup (B \cap (E^l \cup E^h)^c)$

[2] More precisely, letting p_t denote the price of the TESLA stock at time t, it is usually assumed that the process p_t follows a geometric Brownian motion. This implies that the process $\Delta_t = \log(p_t) - \log(p_{t-1})$ is i.i.d.

occurs with probability $\frac{1}{2}$, meaning that L is an unambiguous bet evaluated like a fifty-fifty lottery between \$90 and \$0. In contrast, since according to ℓ_1^∞ both E^h and $B \cap (E^l \cup E^h)^c$ occur with probability $\frac{1}{2}$, while according to ℓ_2^∞ neither occurs, whether H will pay more than \$0 is ambiguous. Since $\alpha * 0 + (1-\alpha)(\frac{1}{2}100 + \frac{1}{2}90) = 23.75 < 45 = \frac{1}{2}90 + \frac{1}{2}0$, the individual prefers to accept the lower payoff of \$90 on E^l instead of \$100 on E^h in exchange for this decrease in ambiguity.

The previous example illustrated a way in which the smooth ambiguity model is more restrictive than the α-MEU model when applied to bets depending on long-run frequencies. Our next example illustrates the reverse – a way in which the α-MEU model is more restrictive than the smooth ambiguity model, even when applied to bets depending on long-run frequencies. Suppose that the individual has to choose among the following bets:

(i) bet h, where h pays \$100 if 50% to 60% of the daily returns are between 0% and 1% and \$0 otherwise;
(ii) bet m, where m pays \$100 if either 20% to 30% or 50% to 60% of the daily returns are between 0% and 1% and \$0 otherwise.

If an individual follows the α-MEU model as in (3), then $h \succ \$0$ and $\$100 \succ m$ together imply $m \sim h$. That i.i.d. α-MEU forces this indifference can be seen using (3) as follows. Observe that $h \succ \$0$ and (3) imply that the set D must contain at least one measure that assigns a probability between 0.5 and 0.6 to the event that daily returns are between 0% and 1%. Similarly, $\$100 \succ m$ and (3) imply that there is a measure in the set D that assigns a probability not in $[0.2, 0.3] \cup [0.5, 0.6]$ to the event that daily returns are between 0% and 1%. Therefore, the value of (3) must be the same for both h and m – as ℓ varies over D, $\int f d\ell^\infty$ can be as good as \$100 and as bad as \$0 for both $f = h$ and $f = m$. This implied indifference is a special case of an axiom called Relevant Range that we introduce in this paper as part of the characterization of (3). Relevant Range requires indifference between any two acts generating the same range of $\int f d\ell^\infty$ as ℓ varies over the set of what Klibanoff et al. (2014) characterize through preferences as *relevant measures*. To complete the connection with the example, note that in the context of (3), D is exactly the set of such relevant measures (Klibanoff et al., 2014, Theorem 4.1). On the other hand, the smooth ambiguity model as in (4) permits $h \succ \$0$ and $\$100 \succ m$ and $m \succ h$. For an example, let μ assign positive weight to each of ℓ_1, ℓ_2, ℓ_3 with $\ell_1^\infty(E^h) = 1 = \ell_2^\infty(E^l)$ and $\ell_3^\infty(E^h \cup E^l) = 0$ and ϕ be any continuous, increasing function.

Finally, for general acts that do not necessarily involve long-run frequency events, the α-MEU model excludes other types of behavior that the smooth ambiguity model does not. To illustrate, suppose that the individual is told that the urn contains equal numbers of black and white balls. Consider the following bets:

(i) bets a and b, where a pays \$100 if the first draw from the urn is black and \$0 otherwise, while b pays \$100 if the first daily return is between 0% and 1% and \$0 otherwise.
(ii) bets A and B, where A pays \$100,000 with probability $\frac{1}{2}$ and with the remaining probability pays \$100 if the first draw from the urn is black and \$0 otherwise. Similarly, B pays \$100,000 with probability $\frac{1}{2}$ and with the remaining probability pays \$100 if the first daily return is between 0% and 1% and \$0 otherwise.

If $a \succ b$ then the α-MEU model implies $A \succ B$, whereas the smooth ambiguity model allows for the choice reversal $B \succ A$. Note that this reversal is consistent with diminished ambiguity aversion at the higher utility levels under A and B no longer being sufficient to support the unambiguous bet A over the ambiguous bet B. The key property of the α-MEU model that implies constant (both absolute and relative) ambiguity aversion, thus ruling out this behavior reflecting ambiguity aversion changing with utility levels, is the Certainty Independence axiom of Gilboa and Schmeidler (1989) (see Section 4.1 for a statement of this axiom).[3] The smooth ambiguity model need not satisfy constant ambiguity aversion (either absolute or relative) and generally violates Certainty Independence (see e.g., Klibanoff et al., 2005 for discussion in this regard). Baillon and Placido (2019) and Berger and Bosetti (2020) provide experimental evidence on non-constant ambiguity aversion.

An advantage of characterizing these two models in the same framework is that it becomes possible to compare them on their whole domain of preferences. As we will see, the difference between the symmetric versions of the two models is that the α-MEU model satisfies Certainty Independence and Relevant Range, while the smooth ambiguity model need not, and when it is not expected utility, cannot. Conversely, the smooth ambiguity model must satisfy axioms of subjective expected utility when restricted to acts whose payoffs depend only on events based on limiting frequencies, while the α-MEU model need not, and cannot unless it is expected utility for all acts. While Certainty Independence and axioms equivalent to subjective expected utility are familiar from the existing literature, the Relevant Range axiom is novel to this paper.[4]

1.1. Related literature

The most closely related literature consists of those papers that either axiomatize versions of the α-MEU or smooth ambiguity models or axiomatize various preferences under symmetry conditions. Consider, first, papers on the foundations of α-MEU. Ghirardato et al. (2004, Proposition 19) characterize α-MEU under the restriction that the set C appearing in the representation (1) is also the unique set of probability measures appearing in the Bewley (2002) style representation of $\succsim^*$, the largest incomplete sub-relation satisfying the Anscombe-Aumann Independence axiom. However, as shown by Eichberger et al. (2011), when the state space is finite their axioms hold if and only if the preferences are either maxmin or maxmax (i.e., $\alpha = 0$ or 1). Klibanoff et al. (2018, Theorem 4.5) extends this conclusion to the context of the symmetric α-MEU model, where the state space is S^∞ and even S need not be finite. In contrast, our approach allows for the full range of $\alpha \in [0, 1]$, albeit only in symmetric environments. Kopylov (2003, Theorem 2.4) characterizes α-MEU under the restriction that the set C appearing in the representation (1) is also the set of probability measures, $\mathcal{M}_0$, that, when used in an expected utility representation, generate preferences agreeing with the restriction of $\succsim$ to the set of subjectively risky acts. Subjectively risky acts are those acts h such that, for all acts f, g, and all $\lambda \in (0, 1)$,

$$f \succsim g \Longleftrightarrow \lambda f + (1-\lambda)h \succsim \lambda g + (1-\lambda)h$$

[3] Note that constant ambiguity aversion here is not referring to the fact that α is constant (i.e., the same across all acts) in an α-MEU representation. In fact, Certainty Independence (and thus constant ambiguity aversion) holds for a much broader class of preferences that includes α-MEU. Ghirardato et al. (2004) call this class the Invariant Biseparable preferences and show they correspond to a type of α-MEU functional where α varies across acts (see also Amarante, 2009). See Grant and Polak (2013) for a weakening of Certainty Independence that they show corresponds to constant absolute (as opposed to constant relative) ambiguity aversion.

[4] See the comparison with axioms from Ghirardato et al. (2004) and from Kopylov (2003) in Section 4.1.

as is required by the Anscombe-Aumann Independence axiom. This restriction has bite, in general. However, we show (see Appendix A.4) that the α-MEU preferences we axiomatize also satisfy Kopylov's (2003) axioms, implying that under preference symmetry the requirement that the set C equal $\mathcal{M}_0$ is unrestrictive. Gul and Pesendorfer (2015) axiomatize a different special case of α-MEU. In their model, there exists a σ-algebra $\mathcal{E}$ and a prior μ defined on $\mathcal{E}$ such that preferences are represented by

$$U(f) = \alpha \min_{\pi \in \Pi_\mu} \int u(f(s))d\pi(s) + (1-\alpha) \max_{\pi \in \Pi_\mu} \int u(f(s))d\pi(s),$$

where Π_μ is the set of *all* probability measures that agree with μ on $\mathcal{E}$. Their interpretation is that the individual is completely ignorant about all events that are not in $\mathcal{E}$ and has no ambiguity about events in $\mathcal{E}$. Such sets Π_μ differ from the sets of measures appearing in our i.i.d. α-MEU model. Chateauneuf et al. (2007) axiomatize a special case of Choquet expected utility that evaluates each act according to a convex combination of the least favorable prize, the most favorable prize and expected utility with respect to a fixed probability. Hill (2019) explores a generalization of the α-MEU model in which the individual considers a convex combination of general uncertainty averse and uncertainty loving preferences. Arrow and Hurwicz (1972) and Cohen and Jaffray (1980) study decision making under complete ignorance. They axiomatize criteria ranking acts based only on the worst and best payoff.

Next, we turn to the foundations of the smooth ambiguity model. Our smooth ambiguity model characterizations (Theorems 3 and 4) are entirely in terms of preferences over acts, but impose preference symmetry. An advantage relative to that in Klibanoff et al. (2005) is that their second order acts are not required. Two advantages relative to Seo (2009) are that failure to reduce objective compound lotteries is no longer implied by non-neutral attitudes to ambiguity, and that, given the vN-M utility function u, the function ϕ that models attitudes toward ambiguity is uniquely identified. Minardi and Savochkin (2017) characterize the special case of the smooth ambiguity model where ϕ has a negative exponential form. Al-Najjar and de Castro (2014) and Cerreia-Vioglio et al. (2013) use versions of symmetry, as we do, to characterize symmetric versions of the smooth ambiguity model. Compared to Al-Najjar and de Castro (2014, Theorem 7), our results are more detailed and more clearly link the representation to a set of preference axioms.[5] The primary contrast with Cerreia-Vioglio et al. (2013, Theorem 6) is that they take the set of probabilities in the support of μ as a primitive, while we derive them from preferences. Recently, Denti and Pomatto (2020) characterize a version of the smooth ambiguity model in which the probability measures in the support of μ are *identifiable*. A set $\mathcal{P}$ of probability measures is identifiable if there is a function k mapping states in Ω to probability measures over Ω such that $k(\omega)(\{\omega' \in \Omega : k(\omega') = k(\omega)\}) = 1$ for all $\omega \in \Omega$ such that $k(\omega) \in \mathcal{P}$. Observe that any subset of i.i.d. measures is identifiable using the function k that associates each state with the i.i.d. measure corresponding to the limiting frequency of that state. Thus the smooth ambiguity model characterized in this paper is a specialization of that in Denti and Pomatto (2020). However, they neither characterize this specialization nor address α-MEU.

[5] First, they do not specify which expected utility axioms the preference has to satisfy for acts measurable with respect to long-run frequency events. As shown in our Theorems 3 and 4, different sets of expected utility axioms not only have different implications for restrictions on the prior μ but also for the function ϕ. Second, they assume that the support of μ contains only countably additive measures. They do not specify the behavioral content of this assumption. We show that the corresponding behavioral property is monotone continuity of $\succsim^*$. Finally, since μ itself is countably additive in their representation, one can show that their continuity axiom on preferences implies that ϕ must be continuous in their representation, but this restriction is not reflected in their analysis or result.

Finally, our paper builds on the literature using preference symmetry conditions to explore ambiguity. In particular, Klibanoff et al. (2014) forms the starting point of our analysis. In turn, it is part of a broader literature generalizing the approach to symmetry pioneered by de Finetti (1937) and Hewitt and Savage (1955). Some of the most relevant references here include Epstein and Seo (2010, 2011), the aforementioned Al-Najjar and de Castro (2014) and Cerreia-Vioglio et al. (2013), and Klibanoff et al. (2018).

1.2. Organization of the paper

Section 2 introduces notation and the main choice-theoretic objects used in the paper. In section 3 we recall a useful result from Klibanoff et al. (2014) that constitutes the starting point of our analysis. Section 4 contains our main results. Section 5 has some discussion and concludes. Section A is an appendix containing all the proofs and some additional results.

2. Setting and notation

We borrow our notation from Klibanoff et al. (2014). Consider a compact metric space S. The state space is given by $\Omega = S^{\infty}$, with generic element $\omega = (\omega_1, \omega_2, ...)$. Observe that, by well-known results, Ω is also a compact metric space. Denote by Σ_i the Borel σ-algebra on the i-th copy of S, and by Σ the product σ-algebra on S^{∞}. Let X be the set of lotteries (i.e., finite support probability measures on an outcome space Z). An act is a simple Anscombe-Aumann act, a measurable $f : S^{\infty} \to X$ having finite range (i.e., $f(S^{\infty})$ is finite). The set of acts is denoted by $\mathcal{F}$, and $\succsim$ is a binary relation on $\mathcal{F} \times \mathcal{F}$ ($\sim$ and $\succ$ denote the symmetric and asymmetric part, respectively). As usual, we identify a constant act with the element of X it yields.

Denote with Π the set of all finite permutations on $\{1, 2, ...\}$ i.e., all one-to-one and onto functions $\pi : \{1, 2, ...\} \to \{1, 2, ...\}$ such that $\pi(i) = i$ for all but finitely many $i \in \{1, 2, ...\}$. For $\pi \in \Pi$, let $\pi\omega = \left(\omega_{\pi(1)}, \omega_{\pi(2)}, ...\right)$ and $(\pi f)(\omega) = f(\pi\omega)$.

For any topological space Y, let $\Delta(Y)$ denote the set of (countably additive) Borel probability measures on Y. $ba(Y)$ is the set of finitely additive bounded real-valued set functions on Y, and $ba_+^1(Y)$ the set of non-negative probability charges in $ba(Y)$. A measure $p \in \Delta(S^{\infty})$ is called symmetric if the order doesn't matter, i.e., $p(A) = p(\pi A)$ for all $\pi \in \Pi$, where $\pi A = \{\pi\omega : \omega \in A\}$. Denote by ℓ^{∞} the i.i.d. measure with the marginal $\ell \in \Delta(S)$. Define $\int_{S^{\infty}} f dp \in X$ by $\left(\int_{S^{\infty}} f dp\right)(B) = \left(\int_{S^{\infty}} f(\omega)(B)\, dp(\omega)\right)$. (Since f is simple, this is well-defined.) We endow $\Delta(S)$, $\Delta(\Delta(S))$ and $\Delta(S^{\infty})$ with the relative weak* topology.[6] For a set $D \subseteq \Delta(S)$, $\overline{D}$ denotes the closure of D in the relative weak* topology.

Fix $x_*, x^* \in X$ such that $x^* \succ x_*$. For any event $A \in \Sigma$, 1_A denotes the act giving x^* on A and x_* otherwise. Informally, this is a bet on A. A finite cylinder event $A \in \Sigma$ is any event of the form $\{\omega : \omega_i \in A_i \text{ for } i = 1, ..., n\}$ for $A_i \in \Sigma_i$ and some finite n. More generally, given $f, g \in \mathcal{F}$ and $A \in \Sigma$, fAg denotes the act that yields $f(s)$ for $s \in A$ and $g(s)$ for $s \notin A$. For a measurable partition $(A_i)_i^m$ of S and lotteries $(x_i)_{i=1}^m$, we denote with $f = \sum_{i=1}^m x_i 1_{A_i}$ the act that yields x_i for $s \in A_i$. An event $A \in \Sigma$ is *null* if $fAg \sim g$ for all acts $f, g \in \mathcal{F}$.

The support of a probability measure $m \in \Delta(\Delta(S))$, denoted $\operatorname{supp} m$, is a relative weak* closed set such that $m\left((\operatorname{supp} m)^c\right) = 0$ and if $G \cap \operatorname{supp} m \neq \emptyset$ for relative weak* open G,

[6] To see what this is, consider, for example, $\Delta(S)$. The relative weak* topology on $\Delta(S)$ is the collection of sets $V \cap \Delta(S)$ for weak* open $V \subseteq ba(S)$, where the weak* topology on $ba(S)$ is the weakest topology for which all functions $\ell \longmapsto \int \psi d\ell$ are continuous for all bounded measurable ψ on S.

$m(G \cap \operatorname{supp} m) > 0$. Let $\Psi_n(\omega) \in \Delta(S)$ denote the empirical frequency operator defined by $\Psi_n(\omega)(A) = \frac{1}{n}\sum_{t=1}^{n} I(\omega_t \in A)$ for each event A in S. Define the limiting frequency operator Ψ by $\Psi(\omega)(A) = \lim_n \Psi_n(\omega)(A)$ if the limit exists and 0 otherwise. Also, to map given limiting frequencies or sets of limiting frequencies to events in S^∞, we consider the natural inverses $\Psi^{-1}(\ell) = \{\omega : \Psi(\omega) = \ell\}$ and $\Psi^{-1}(L) = \{\omega : \Psi(\omega) \in L\}$ for $\ell \in \Delta(S)$ and $L \subseteq \Delta(S)$.

Finally, we denote by Σ^Δ the σ-algebra generated by the open (in the weak* topology) sets in $\Delta(S)$. Let Σ^Ψ denote the σ-algebra generated by the collection of sets $\{\Psi^{-1}(L) : L \in \Sigma^\Delta\}$. Denote by $\mathcal{F}^\Psi$ the set of simple acts measurable with respect to Σ^Ψ.

3. Symmetry and Relevance

We next recall axioms and a key definition from Klibanoff et al. (2014). Consider the following axioms on $\succsim$ which will be common to both models. The first five are standard axioms in an Anscombe-Aumann framework.

Axiom 1. $\succsim$ is complete and transitive.

Axiom 2 *(Monotonicity).* If $f(\omega) \succsim g(\omega)$ for all $\omega \in S^\infty$, $f \succsim g$.

Axiom 3 *(Risk Independence).* For all $x, x', x'' \in X$ and $\alpha \in (0, 1)$, $x \succsim x'$ if and only if $\alpha x + (1-\alpha)x'' \succsim \alpha x' + (1-\alpha)x''$.

Axiom 4 *(Non-triviality).* There exist $x, y \in X$ such that $x \succ y$.

Axiom 5 *(Mixture Continuity).* For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0, 1] : \lambda f + (1-\lambda)g \succsim h\}$ and $\{\lambda \in [0, 1] : h \succsim \lambda f + (1-\lambda)g\}$ are closed in $[0, 1]$.

The last two shared axioms are written in terms of the binary relation $\succsim^*$ derived from $\succsim$ as follows (see Ghirardato et al., 2004): for every $f, g \in \mathcal{F}$,

$$f \succsim^* g \quad \text{if} \quad \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h \text{ for all } \alpha \in [0, 1] \text{ and } h \in \mathcal{F}.$$

The next axiom says that the coordinates of S^∞ are viewed as interchangeable. Event Symmetry is the main condition that enables our representation results. Thanks to this assumption, acts in $\mathcal{F}^\Psi$ will be able to play the role that second-order acts did in Klibanoff et al.'s (2005, Theorems 1 and 4) axiomatization of the smooth ambiguity model, and the set $D \subseteq \Delta(S)$ in the i.i.d. α-MEU representation is able to be uniquely identified without restrictions on α (see Section 4).

Axiom 6 *(Event Symmetry).* For all finite cylinder events $A \in \Sigma$ and finite permutations $\pi \in \Pi$, $1_A \sim^* 1_{\pi A}$.

A natural notion of symmetry, as expressed through preferences, is that the decision maker is always indifferent between betting on an event and betting on its permutation. The use of the term "always" here means at least that this preference should hold no matter what other act the individual faces in combination with the bet. In an Anscombe-Aumann framework such as ours, this may be expressed by the statement $\alpha 1_A + (1-\alpha)h \sim \alpha 1_{\pi A} + (1-\alpha)h$ for all $\alpha \in [0, 1]$ and all acts h, which is precisely $1_A \sim^* 1_{\pi A}$. For preferences satisfying the usual independence axiom, $1_A \sim^* 1_{\pi A}$ is equivalent to $1_A \sim 1_{\pi A}$. As a main goal of our analysis is to accommodate preferences that violate independence, we cannot substitute the former with the latter.

The remaining shared axiom is a continuity requirement on $\succsim^*$:

Axiom 7 *(Monotone Continuity of $\succsim^*$).* For all $x, x', x'' \in X$, if $x' \succ x''$ and events $(A_n)_{n=1}^{\infty}$ are such that $A_{i-1} \supseteq A_i$ for every i and $\cap_{n=1}^{\infty} A_n = \emptyset$, then $x' \succsim^* x A_n x''$ for some n.

In addition to these axioms, we also borrow from Klibanoff et al. (2014) a key definition. Define a *relevant measure* as a marginal distribution, ℓ, on S that matters for preferences in the following sense: For each open set of marginal distributions, L, containing ℓ, we can find two acts, f and g, that yield the same distribution over outcomes as each other under all i.i.d. distributions generated by marginals not in L and yet the individual strictly prefers f over g.

Let $\mathcal{O}_\ell$ denote the set of open subsets of $\Delta(S)$ that contains ℓ. The use of these open neighborhoods in the definition is required only because $\Delta(S)$ is infinite. The formal definition is:

Definition 1. A measure $\ell \in \Delta(S)$ is *relevant* according to preferences $\succsim$ if for any $L \in \mathcal{O}_\ell$, there are $f, g \in \mathcal{F}$ such that $f \nsim g$ and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\hat{\ell} \in \Delta(S)\backslash L$. The set of relevant measures for preferences $\succsim$ is denoted by $R(\succsim)$.

Given $\succsim$ satisfying Axioms 1–7, $R(\succsim)$ is unique. $R(\succsim)$ is endogenous in that it is defined from, and hence varies with, the primitive, $\succsim$.

To better understand the relationship between preferences and the corresponding $R(\succsim)$, consider the following result showing that a marginal $\ell \in \Delta(S)$ is a relevant measure if and only if, for each open neighborhood containing it, the corresponding limiting frequency event is non-null according to preferences. In reading it, recall that, for $A \subseteq \Delta(S)$, $\Psi^{-1}(A)$ is the event that limiting frequencies over S lie in A.

Theorem 1. (Klibanoff et al. 2014, Theorem 3.2) *Assume $\succsim$ satisfies Axioms 1–7. For $\ell \in \Delta(S)$, $\ell \notin R(\succsim)$ if and only if, for some $L \in \mathcal{O}_\ell$, $\Psi^{-1}(L)$ is a null event according to $\succsim$. Moreover, $R(\succsim)$ is closed.*

When $R(\succsim)$ is finite, the same result holds without the use of neighborhoods, i.e., $\Psi^{-1}(\ell)$ is null according to $\succsim$ if and only if $\ell \notin R(\succsim)$. Theorem 1 justifies thinking of $R(\succsim)$ as the unique set of marginals subjectively viewed as possible, since the individual behaves as if only those outside of $R(\succsim)$ are impossible. Note the role of Axioms 1–7 (especially the Event Symmetry axiom): they allow marginals over S to be identified behaviorally with (limiting frequency) events in S^∞. Given that perceived ambiguity is subjective uncertainty about probability assignments, under Axioms 1–7 the relevant measures are the probability assignments revealed to be in the support of that uncertainty. In other words, relevant measures are those corresponding to non-null limiting frequency events.

In the next section we will provide axioms that, together with Axioms 1–7, are equivalent to the α-MEU and smooth ambiguity representations as in (3) and (4).

4. Using Event Symmetry and Relevance to provide foundations for the two decision models

In this section, we characterize the α-MEU and smooth ambiguity models under our symmetry and continuity assumptions.

4.1. α-MEU model

We will show that under Axioms 1–7, α-MEU is what results from strengthening Risk Independence to Gilboa and Schmeidler's (1989) Certainty Independence (stated below) and adding an axiom making use of the following set:

$$C^*(f) \equiv \{x \in X : x \succsim \int f d\ell^\infty \text{ for some } \ell \in R(\succsim) \\ \text{and } \int f d\ell^\infty \succsim x \text{ for some } \ell \in R(\succsim)\}.$$

The set $C^*(f)$ consists of the lotteries that (in terms of preference) lie in the range of lotteries induced by f under the i.i.d. measures generated from relevant measures (i.e., between the best and worst lotteries formed by using ℓ^∞ for $\ell \in R(\succsim)$ to weight the outcomes of f). The new axiom needed to characterize α-MEU says that if two acts have the same sets C^* then the individual must be indifferent between them.

Axiom 8 *(Relevant Range).* For all $f, g \in \mathcal{F}$, $C^*(f) = C^*(g)$ implies $f \sim g$.

We also need the following strengthening of Risk Independence, introduced by Gilboa and Schmeidler (1989),

Axiom 9 *(Certainty Independence).* For all $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0, 1)$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x$.

Notice that Certainty Independence remains weaker than the following standard Independence axiom (which would lead to SEU):

Axiom 10 *(Independence).* For all $f, g, h \in \mathcal{F}$, and $\alpha \in (0, 1)$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

The next result shows that Axioms 1–7, when strengthened by adding Relevant Range and replacing Risk Independence with Certainty Independence, characterize the α-MEU model in (3).

Theorem 2. *$\succsim$ satisfies Relevant Range and Axioms 1–7 with Risk Independence replaced by Certainty Independence if and only if $R(\succsim)$ is finite and there is a non-constant vNM utility function u and an $\alpha \in [0, 1]$ such that*

$$V(f) \equiv \alpha \min_{p \in \{\ell^\infty : \ell \in R(\succsim)\}} \int u(f)\, dp + (1-\alpha) \max_{p \in \{\ell^\infty : \ell \in R(\succsim)\}} \int u(f)\, dp, \tag{5}$$

represents $\succsim$.

Furthermore, for any non-constant vNM utility function $\hat{u}$, $\hat{\alpha} \in [0, 1]$ and finite set $D \subseteq \Delta(S)$, the preferences $\hat{\succsim}$ represented by

$$\hat{V}(f) \equiv \hat{\alpha} \min_{p \in \{\ell^\infty : \ell \in D\}} \int \hat{u}(f)\, dp + (1 - \hat{\alpha}) \max_{p \in \{\ell^\infty : \ell \in D\}} \int \hat{u}(f)\, dp, \tag{6}$$

satisfy Relevant Range and Axioms 1–7 with Risk Independence replaced by Certainty Independence. Moreover, $R(\hat{\succsim}) = D$. Finally, two functionals of the form in (6) represent the same preferences if and only if they have the same set D, the utility functions are related by a positive affine transformation, and, if D is non-singleton, they have the same $\hat{\alpha}$.

This characterizes the α-MEU model, albeit limited to symmetric environments and finitely generated sets of countably additive measures.[7] Our uniqueness results ensure that the set of measures and α are meaningful. The representation in (5) shows how the set of measures in i.i.d. α-MEU is related to the endogenous set of relevant measures $R(\succsim)$. This way of writing the representation is analogous to the α-MEU representations in Ghirardato et al. (2004, Proposition 19) and Kopylov (2003, Theorem 2.4) in that they also write the set of measures in terms of an endogenous construct – the Bewley set C in the case of Ghirardato et al. (2004) and the set $\mathcal{M}_0$ in the case of Kopylov (2003) (recall the description of these sets from Section 1.1). Any representation in which the set of measures is tied to such an endogenous construct raises the question of which actual sets of measures and parameters α are consistent with at least some preference satisfying the given axioms. The contribution of the second part of Theorem 2 with representation (6) is to show that for i.i.d. α-MEU the fact that the set of measures must be generated by $R(\succsim)$ is unrestrictive – *any* finite set of marginals, D, together with *any* $\alpha \in [0, 1]$ generates a preference satisfying the axioms, making it clear that the entire class of i.i.d. α-MEU representations (3) is what the axioms characterize.

In applications, it is often desirable to model an agent who has some particular finite set of probability measures in mind along with a particular α. It follows from Theorem 2 that any combination of the two is consistent with the axioms, and so, in that sense, they are indeed free to be separately specified. In contrast, Ghirardato et al. (2004) and Kopylov (2003) do not provide results analogous to the second part of our Theorem 2. As was discussed in Section 1.1, in fact, under either symmetry or a finite state space, Ghirardato et al.'s (2004) characterization is limited to cases where α is 0 or 1. Kopylov (2003, p. 91) pointed out that it is easy to find combinations of sets of measures and αs violating his axioms. As an additional contribution of our analysis (see Appendix A.4), we show that under symmetry an analogous result does hold for Kopylov's theory: all i.i.d. α-MEU representations satisfy Kopylov's axioms.

It is worth noting that Ghirardato et al.'s (2004) Axiom 7 is similar to Relevant Range except that it uses the range generated by measures in the Bewley set $C(\succsim)$ rather than measures in $R(\succsim)$. Importantly, while $R(\succsim)$ does not depend on α, the set $C(\succsim)$ does. That $R(\succsim)$ is independent of α can be seen from the facts that (a) whether an event is non-null is independent of α, and (b) as shown in Theorem 1, $R(\succsim)$ is fully determined by which limiting-frequency events are non-null according to $\succsim$. In contrast, the Bewley set $C(\succsim)$ in Ghirardato et al.'s (2004) Axiom 7 uses not just preference information about on which events utility has value (i.e., which events are non-null), but also information about the relative magnitudes of those valuations (i.e., how

[7] The proof reveals that finiteness of $R(\succsim)$ results from a tension between Monotone Continuity of $\succsim^*$ (which is the main force ensuring countable additivity) and the conjunction of Relevant Range and Certainty Independence (which are the main drivers ensuring the α-MEU form).

preferences trade-off utility across different non-null events). Mathematically, according to their Theorem 14, the Bewley set $C(\succsim)$ is the Clark differential of the representing functional at the constant utility 0. Since this Clark differential of the α-MEU functional depends on α, so must $C(\succsim)$.

Given an i.i.d. α-MEU representation, preferences will satisfy Relevant Range with respect to the set of measures appearing in the representation. Except when α is 0 or 1 or the set of measures in the representation is a singleton, the Bewley set $C(\succsim)$ will not be equal to the set of measures appearing in the α-MEU model, and thus Ghirardato et al.'s (2004) Axiom 7 will not hold. Combining Ghirardato et al. (2004, Proposition 19), our Theorem 2, and Theorems 4.2 and 4.5 in Klibanoff et al. (2018), if Relevant Range were replaced by their Axiom 7, then either α is 0 or 1 or the set of measures in the representation is a singleton. That is, under their Axiom 7, $\succsim$ must be MEU or max-max EU, in which cases the earlier characterization results of Gilboa and Schmeidler (1989) already apply. In this sense, our result shows that in a symmetric environment, the difference between the α-MEU model and the union of the MEU and max-max EU models is exactly the difference between Relevant Range and Ghirardato et al.'s (2004) Axiom 7. Recall that the key role of symmetry in the process is in allowing for the relevant measures to be identified from non-nullity of events in the state space Ω.

Another counterpart to Relevant Range is the Partial Ignorance outside of the Subjectively Risky Acts axiom in Kopylov (2003) (recall his notion of subjectively risky act from our discussion in Section 1.1). The axiom itself is complex to state, but has the following spirit: if the ranking over subjectively risky acts plus the implications of Completeness, Transitivity, Monotonicity and Certainty Independence do not force one to conclude that $f \succ g$ or $g \succ f$, then the axiom requires that $f \sim g$.

We observe that the Relevant Range axiom can also be related to two key axioms (called Property B and Property C) in Arrow and Hurwicz (1972) (that they attribute to Chernoff (1954)). Property B states that relabeling of actions and states of nature should be deemed irrelevant by the individual. Property C requires that for a given decision problem, if a state gives the same payoff as another state for every action in the decision problem, then the state can be removed without influencing the optimal action for the decision problem. Together with a monotonicity assumption, these two properties imply that if two actions have the same range of payoffs then they should be deemed equivalent by the individual. As they argue, this last property is a way to model decision making under "complete ignorance", i.e. that there is no a priori information available which gives any state of nature a distinguished position. Similarly, our Relevant Range axiom reflects the idea that beyond whether a frequency is relevant or not there is "complete ignorance" over such relevant frequencies.

Finally, it is useful to know which i.i.d. α-MEU preferences are also subjective expected utility preferences (i.e., also satisfy the Independence axiom). The following result gives the overlap:

Proposition 1. *Consider a preference relation $\succsim$ represented by (6). The preference satisfies Independence if and only if either D is a singleton or D has two elements and $\alpha = \frac{1}{2}$.*

4.2. Smooth ambiguity model

We provide two different foundations for the smooth ambiguity model building on Axioms 1–7. The main idea is to additionally impose expected utility axioms on a subclass of acts, those measurable with respect to events defined by empirical frequency limits, i.e., the acts in $\mathcal{F}^{\Psi}$. Recall that $\mathcal{F}^{\Psi}$ denotes the set of simple acts that are measurable with respect to events in

Σ^{Ψ}, i.e., long-run frequency events. Before proceeding, it is important to point out that imposing expected utility axioms on $\mathcal{F}^{\Psi}$ does *not* imply that the individual views events in Σ^{Ψ} as unambiguous nor that the individual is ambiguity neutral when evaluating such acts.[8] A final, more technical addition is a Cauchy continuity axiom important in ensuring the existence of a monotonic and norm-continuous extension of preferences from the simple acts, $\mathcal{F}$, to the bounded acts, $\hat{\mathcal{F}}$, which is used in connecting preferences on $\mathcal{F}^{\Psi}$ to preferences on all simple acts. What differentiates the two characterizations is the axioms imposed on preferences restricted to $\mathcal{F}^{\Psi}$. The first characterization uses the famous axioms from Savage (1954) which have great familiarity, simplicity and transparency, at the cost of requiring that the measure μ in the representation is non-atomic.[9] The second characterization allows for the important possibility of μ having general finite or infinite support by substituting axioms from Wakker (1989, Theorem V.6.1), the main one being a tradeoff consistency axiom, for those of Savage.

Start by considering the following version of Savage's postulates for acts in $\mathcal{F}^{\Psi}$.[10]

P2 For every $f, g, h, h' \in \mathcal{F}^{\Psi}$ and $A \in \Sigma^{\Psi}$,

$$fAh \succsim gAh \implies fAh' \succsim gAh'.$$

As usual, $A \in \Sigma^{\Psi}$ is *null* if for every $f, g \in \mathcal{F}^{\Psi}$, $fAg \sim g$, otherwise it is *non-null*.

P3 For every $x, y \in X$, $f, g \in \mathcal{F}^{\Psi}$, and non-null $A \in \Sigma^{\Psi}$,

$$x \succ y \iff xAf \succ yAg.$$

P4 For every $A, B \in \Sigma^{\Psi}$ and $x, y, x', y' \in X$ such that $x \succ y$ and $x' \succ y'$,

$$xAy \succsim xBy \implies x'Ay' \succsim x'By'.$$

P6 For every $f, g \in \mathcal{F}^{\Psi}$ and $x \in X$ such that $g \succ f$, there exists a Σ^{Ψ}-measurable partition $(A_i)_{i=1}^{n}$ of Ω such that for every $i = 1, \ldots, n$, $g \succ x_{A_i} f$ and $x_{A_i} g \succ f$.

As in Savage, a key implication of P6 is that μ is non-atomic. Additionally, the following continuity axiom is used to ensure countable additivity of μ and continuity of ϕ. Given a sequence $(f)_{n=1}^{\infty}$ of acts, write $f_n \to f$ if for every act g, $g \succ f$ implies that there exists N such that $n \geq N \implies g \succ f_n$ and $f \succ g$ implies that there exists N' such that $n \geq N' \implies f_n \succ g$.

Axiom 11 *(Pointwise Continuity).* For every sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{F}^{\Psi}$ such that for some $x, y \in X$ $x \succsim f_n(\omega) \succsim y$ for every $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$ for every $\omega \in \Omega$ implies $f_n \to f$.

Our final axiom is the Cauchy continuity requirement. Let $\hat{\mathcal{F}}$ denote the set of all bounded and measurable functions from Ω to X.[11] Ghirardato and Siniscalchi (2010) propose a notion

[8] We discuss and illustrate this in Section 5.2.

[9] In Appendix A.7, we show that by weakening Savage's P6, the same approach can be extended to allow μ that are only partially non-atomic – in particular, μ may assign up to half its weight to atoms. Note that this still completely rules out discrete measures or measures with finite support.

[10] Axioms 1 and 4 already provide Savage's postulates P1 and P5, and so we do not repeat those here.

[11] More precisely, $\hat{\mathcal{F}}$ is the collection of functions $f : \Omega \to X$ that satisfy the following two properties:

(i) for all $x \in X$, $\{\omega : f(\omega) \succ x\} \in \Sigma$; and
(ii) there exist $x, y \in X$ such that $x \succsim f(\omega) \succsim y$ for all $\omega \in \Omega$.

of convergence that they show corresponds to sup-norm convergence in the space of utility acts. Following them, we say $f_k \in \hat{\mathcal{F}}$ *norm-converges* to $f \in \hat{\mathcal{F}}$ if for all $x, y \in X$ with $x \succ y$, there exists K such that $k \geq K$ implies for all $\omega \in \Omega$

$$\tfrac{1}{2} f(\omega) + \tfrac{1}{2} y \prec \tfrac{1}{2} f_k(\omega) + \tfrac{1}{2} x \text{ and } \tfrac{1}{2} f_k(\omega) + \tfrac{1}{2} y \prec \tfrac{1}{2} f(\omega) + \tfrac{1}{2} x.$$

Ghirardato and Siniscalchi (2010) propose the following continuity condition using norm-convergence:

Axiom 12 *(Cauchy Continuity).* Consider sequences $f_k \in \mathcal{F}$, $x_k \in X$ such that f_k norm-converges to $f \in \hat{\mathcal{F}}$. If $f_k \sim x_k$ for all k, then there exists $x \in X$ such that x_k norm-converges to x.

We are ready to state our first representation theorem for the smooth ambiguity model.

Theorem 3. *$\succsim$ satisfies Axioms 1–7, P2-P4, P6, Pointwise Continuity and Cauchy Continuity if and only if there is a non-constant vNM utility function $u : X \to \mathbb{R}$, a strictly increasing continuous function $\phi : u(X) \to \mathbb{R}$ such that there are $m, M > 0$ with $m|x - y| \leq |\phi(x) - \phi(y)| \leq M|x - y|$ for every $x, y \in u(X)$ and a non-atomic Borel probability measure $\mu \in \Delta(\Delta(S))$ such that*

$$U(f) = \int_{\Delta(S)} \phi\left(\int u(f) d\ell^{\infty}\right) \mu(\ell),$$

represents $\succsim$. Moreover, μ is unique, $R(\succsim) = \operatorname{supp} \mu$, u is unique up to a positive affine transformation, and, given a normalization of u, ϕ is unique up to positive affine transformations.

Note that the Lipschitz style restriction $m|x - y| \leq |\phi(x) - \phi(y)| \leq M|x - y|$ is only needed to guarantee Monotone Continuity of $\succsim^*$.[12]

Next we provide the axiomatization relying on Wakker's Tradeoff Consistency and S-continuity axioms. To apply Wakker (1989, Theorem V.6.1) we need to specify a topology on X. For this purpose, assume that Z is a metric space that is complete and separable.[13] Endow X with the weak convergence (wc) topology. The wc topology on X is the weakest topology for which all functions $x \longmapsto \int \psi dx$ are continuous for all bounded continuous ψ on Z. Also note that a sequence $x_n \in X$ converges to $x \in X$ under the wc topology if and only if $\int \psi dx_n \to \int \psi dx$ for all bounded continuous ψ on Z. Under this topology, X is a connected topological space.

Axiom 13 *(Tradeoff Consistency).* There are no non-null events $A, B \in \Sigma^{\Psi}$, consequences $w, x, y, z \in X$ and acts $f, g \in \mathcal{F}^{\Psi}$ such that $xAf \succsim yAg$, $zAf \succsim wAg$, $xBf \succsim yBg$ and $wBg \succ zBf$.

Axiom 14 *(S-continuity).* For every partition of Ω into a finite number of events in Σ^{Ψ}, $(A_i)_{i=1}^m$, and act $f = \sum_{i=1}^m x_i 1_{A_i}$, the sets $\left\{(y_i)_i^m : \sum_{i=1}^m y_i 1_{A_i} \succsim f\right\}$ and $\left\{(y_i)_i^m : f \succsim \sum_{i=1}^m y_i 1_{A_i}\right\}$ are closed in the product topology of X^m.

12 In this respect, we see that any tension between Monotone Continuity of $\succsim^*$ and the expected utility axioms imposed on acts measurable with respect to limiting frequency events can be resolved through conditions on ϕ rather than by being forced to limit the richness of the set of relevant measures as was the case for α-MEU in Theorem 2.

13 Recall that X is the set of all lotteries over the set Z.

Our representation theorem for the smooth ambiguity model allowing for general μ is:

Theorem 4. *$\succsim$ satisfies Axioms 1–7, S-continuity, Cauchy Continuity and Tradeoff Consistency if and only if there is a non-constant wc-continuous vNM utility function u, a strictly increasing continuous function $\phi : u(X) \to \mathbb{R}$ and a Borel probability measure $\mu \in \Delta(\Delta(S))$ such that*

$$U(f) = \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)\, d\ell^\infty\right) d\mu(\ell), \tag{7}$$

represents $\succsim$ and either (i) there are $m, M > 0$ such that $m|a-b| \le |\phi(a) - \phi(b)| \le M|a-b|$ for all $a, b \in u(X)$ or, (ii) $\operatorname{supp}\mu$ is finite. Moreover, μ is unique, $R(\succsim) = \operatorname{supp}\mu$, u is unique up to positive affine transformations, and, given a normalization of u, if $\operatorname{supp}\mu$ is non-singleton, then ϕ is unique up to positive affine transformations.

Note that the restriction that (i) or (ii) holds is solely to ensure Monotone Continuity of $\succsim^*$, and is a combination of the conditions having the same purpose in Theorems 2 and 3.

In both smooth ambiguity representation theorems, μ is uniquely determined by expected utility preferences over "frequency acts" (i.e., Σ^Ψ-measurable acts) and thus, it expresses beliefs over the events in Σ^Ψ in the same sense as the prior in an expected utility representation. Furthermore, the support of μ is exactly the set of relevant measures. Notice that ϕ is unique up to positive affine transformations only *given* a normalization of u. Should one worry that normalization of u is needed to pin down ϕ, and thus to pin down ambiguity attitude? The answer is no. Expected utility preferences over monetary lotteries have their risk aversion as measured by the Arrow-Pratt index depend on the currency used to denominate money. This in no way means that risk attitudes are non-unique. Similarly, the Arrow-Pratt index of ϕ, identified by Klibanoff et al. (2005) as measuring ambiguity attitude, depends on the units used to measure utility, and this does not affect the unique identification of ambiguity attitudes.

Theorems 3 and 4 provide foundations for the smooth ambiguity model using the Event Symmetry requirement. There are close analogies to the smooth ambiguity representation theorems in Klibanoff et al. (2005) and Seo (2009) with the additional assumption that the environment is known to be symmetric. For all these approaches, the key assumptions are (1) conditions equivalent to expected utility over lotteries, (2) conditions equivalent to expected utility over acts in $\mathcal{F}^\Psi$ (resp. second order acts in Klibanoff et al. (2005) and lotteries over acts in Seo (2009)) and (3) Event Symmetry. In particular, Event Symmetry permits the identification of acts in $\mathcal{F}^\Psi$ with maps from probability measures in $\Delta(S)$ to consequences in X. In this sense, Event Symmetry plays the same role as Klibanoff et al.'s (2005) Consistency and Seo's (2009) Dominance. A formal connection between Event Symmetry and these two axioms is discussed in Klibanoff et al. (2018, pp. 37-39). In particular, conditions (vii) and (viii) in their Theorem 3.1 (reported as Theorem 5 in our Appendix A.1) show how Event Symmetry is equivalent in this context to Dominance and Consistency, respectively. Moreover, our representation results show how, in a symmetric setting, objects like second order acts or lotteries over acts can be replaced by particular standard acts related to frequencies. See the discussions in Klibanoff et al. (2005, pp. 1854 and 1856), (2009, p. 937) for the idea that objects like second order acts could, with enough invariance, be replaced by acts based on long run outcomes of repeated trials.

Finally, we provide a counterpart to Proposition 1, and characterize the overlap between the smooth ambiguity preferences in (7) and subjective expected utility preferences:

Proposition 2. *Consider a preference relation $\succsim$ represented by (7). The preference satisfies Independence if and only if either the support of μ is a singleton or ϕ is linear.*

5. Conclusion and discussion

In recent decades, many models have emerged in pure and applied economic theory according to which agents' choices may be sensitive to ambiguity. Several papers have tried to discriminate among various of these models empirically (among many others, see Cubitt et al. (2020) and Baillon and Bleichrodt (2015)). Moreover, symmetry assumptions are often in the background when analyzing such experimental data. The α-MEU and smooth ambiguity models are two popular alternatives used in applications. By axiomatizing these two models in a common framework, our work can help in understanding and discriminating between them. As we have shown, under symmetry the difference between the two models is exactly that the α-MEU model satisfies Certainty Independence and Relevant Range for all acts while the smooth ambiguity model satisfies axioms of subjective expected utility restricted to acts measurable with respect to long-run frequency events. In contrast, when restricted to the latter acts, i.i.d. α-MEU reduces to a representation of preferences under complete ignorance proposed by Hurwicz (1951) (see also Arrow and Hurwicz, 1972) when the state space is taken as equal to the set of relevant measures[14]:

$$V(f) \equiv \alpha \min_{\ell \in R(\succsim)} u(f) + (1-\alpha) \max_{\ell \in R(\succsim)} u(f). \tag{8}$$

The introduction included some thought experiments illustrating aspects of these differences, both for frequency acts and more general acts.

5.1. Bets on frequency limits

Our axiomatizations make heavy use of the infinite product structure of the state space, $S^\infty = S \times S \ldots \times S \ldots$. Because infinitely many experiments cannot be performed, one may argue that the acts in $\mathcal{F}^\Psi$ are not fully operational. First, we note that the practice of using acts that may require infinite data to determine their realization is ubiquitous in decision theory. For example, in Savage's subjective expected utility theory with a continuum of states, observing the realized state will in general require uncountably infinite data. Moreover, we argue that this type of experiment is already effectively operationalized in economics. For example, in the experimental literature that studies learning in games, a focus is to understand whether play converges to a Nash equilibrium (see for example Chen and Gazzale, 2004). This is usually tested by fixing a long time horizon and looking at whether a high number of players repeatedly play the Nash profile. Another example is related to the experimental literature that tests theoretical predictions of bargaining models. Since the main bargaining models adopt an infinite horizon, again in experiments one has to use a long enough time horizon (e.g., see Weg et al., 1990). Both these cases rely on the idea that an infinite horizon model can be approximated with arbitrary precision by one with finite horizon. Such an idea can be captured in our framework as follows. Consider the simple case of coin tossing, i.e. $S = \{H, T\}$ (so that $\Delta(S) = [0, 1]$) and consider finitely many coin tosses, i.e. $\Omega = \{H, T\}^N$ with $N < \infty$. There are two difficulties. First, symmetry is not

[14] Since these acts are measurable with respect to limiting frequency events, $u(f)$ evaluated at $\ell \in \Delta(S)$ is well-defined and equal to $u(f(\Psi^{-1}(\ell)))$.

equivalent to mixture of i.i.d. measures, as discussed by Diaconis (1977); Diaconis and Freedman (1980). Second, $\Psi_N^{-1}(\ell)$ is empty whenever $\ell \in [0,1]$ is not a rational number. However, as shown by Diaconis and Freedman (1980), any symmetric probability can be approximated by a mixture of i.i.d. measures. Furthermore, the event $\Psi_N^{-1}(\ell)$ with ℓ irrational can be approximated with N large enough.

5.2. *Ambiguity of long-run frequency events*

As stated in Section 4.2, the fact that the smooth ambiguity model under symmetry satisfies the axioms of expected utility on $\mathcal{F}^\Psi$ does *not* imply that the individual views events in Σ^Ψ (i.e., long-run frequency events) as unambiguous or treats them as such. Such an observation is related to a phenomenon known as source preference (see Abdellaoui et al. 2011, p. 696 for a discussion of the literature). Sources of uncertainty are groups of events that are generated by the same mechanism of uncertainty. As demonstrated by Chew and Sagi (2008), one can have probabilities *within* sources even when probabilistic sophistication does not hold *between* sources. In this case, even if the smooth ambiguity model satisfies the sure-thing principle for bets on frequency events, this does not mean that bets on such events are treated in the same way as purely risky bets.

To illustrate, consider the following example. Take any non-null long-run frequency event $E \in \Sigma^\Psi$ and assume smooth ambiguity preferences represented as in (4). Let $m = \mu(\Psi^{-1}(E))$ with $0 < m < 1$. Take $x, y \in X$ such that $u(y) = 0$, $u(x) = 1$ and assume that ϕ is strictly concave with $\phi(0) = 0$. The smooth ambiguity model evaluates the bet xEy as $m\phi(1)$, and the bet xE^cy as $(1-m)\phi(1)$. For any $p \in [0,1]$ and $x, y \in X$, denote by $xpy \in X$ the lottery that pays x with probability p and y with probability $1-p$. Now consider the lotteries xpy and $x(1-p)y$. These are evaluated as $\phi(p)$ and $\phi(1-p)$, respectively. Let p^E and p^{E^c} be such that $xp^Ey \sim xEy$ and $xp^{E^c}y \sim xE^cy$. By strict concavity of ϕ, it follows that $p^E < m$ and $m < 1 - p^{E^c}$. In other words, in terms of betting on such frequency events, the decision maker behaves as if his second order belief μ, the subjective belief about the i.i.d. measures in the set $\Psi^{-1}(E)$, matches an interval of probabilities, $\left[p^E, 1-p^{E^c}\right]$, rather than the precise point $m = \mu\left(\Psi^{-1}(E)\right)$, and this interval is wider the greater the ambiguity aversion. For example, if $\phi(x) = \frac{1}{a}(1-e^{-ax})$, then as $a \to \infty$ we have that $p^E \to 0$ and $1 - p^{E^c} \to 1$.[15]

To formalize this point, we apply the preference-based definition of unambiguous events given by Klibanoff et al. (2005, Definition 7). In the present setting, their definition can be translated as follows:

Definition 2. An event $E \subseteq \Omega$ is unambiguous for the preference $\succsim$ with a smooth ambiguity representation if, for each $x, y \in X$ and $p \in [0,1]$ such that $x \succ y$ either $[xEy \succ xpy$ and $yEx \prec ypx]$ or $[xEy \prec xpy$ and $yEx \succ ypx]$ or $[xEy \sim xpy$ and $yEx \sim ypx]$. An event is ambiguous if it is not unambiguous.

For instance, in the previous example we had $p^E < m$ and $m < 1 - p^{E^c}$, so that $xpy \succ xEy$ and $yEx \succ ypx$, which implies that event E is ambiguous. More generally, using arguments from Klibanoff et al. (2012, Section 2.4) it can be shown that all the (non-null and non-universal)

15 Indeed, $p^E = -\log(1 - m(1-e^{-a}))/a$ and $1 - p^{E^c} = (\log(m - e^{-a})(m-1)) + a)/a$.

events concerning the frequencies of ambiguous events will themselves be treated as ambiguous by the smooth ambiguity model under symmetry.

Appendix A. Proofs and additional results

Denote by $B_0(S, K)$ the set of simple functions defined on S with range contained in an interval K. The set $B_0(\Delta(S), u(X))$ is defined analogously, where $\Delta(S)$ is endowed with the Borel σ-algebra generated by the weak topology.

A.1. Preliminaries

We first report a result from Klibanoff et al. (2018) that will be useful in the main proofs. This result shows that Event Symmetry relates quite closely to a variety of other conditions from the literature, including strengthenings of de Finetti's (1937) Exchangeability, Hewitt and Savage's (1955) Symmetry, of Seo's (2009) Dominance and of Klibanoff et al.'s (2005) Consistency. One of those conditions (condition (viii) below) requires some additional definitions.

Definition 3. For $f \in \mathcal{F}$, f^{Ψ} is the (not necessarily simple) act uniquely defined as follows:

$$f^{\Psi}(\omega) = \begin{cases} \int_{S^\infty} f d\ell^\infty & \text{if } \ell = \Psi(\omega) \in \Delta(S), \\ \delta_{x^*} & \omega \in \{\omega : \Psi(\omega) \text{ is not defined}\}. \end{cases}$$

Note this definition associates with each act f an act f^{Ψ} that, for each event $\{\omega : \Psi(\omega) = \ell\}$ corresponding to the limiting frequencies generated by ℓ, yields the lottery generated by f under the assumption that the i.i.d. process ℓ^∞ governs the realization of the state.

Since f^{Ψ} need not be simple, but is an element of the space $\hat{\mathcal{F}}$ (defined in Section 4.2) of all bounded and measurable functions from Ω to X, it is necessary to consider extending $\succsim$ to $\hat{\mathcal{F}}$. In particular, we consider extensions continuous in the following sense: $\hat{\succsim}$ on $\hat{\mathcal{F}}$ satisfies *Norm Continuity* if $f \hat{\succsim} g$ whenever $f_k \hat{\succsim} g_k$ for all $k = 1, 2, ...$ and f_k and g_k norm-converge to f and g respectively.

Theorem 5. (Klibanoff et al., 2018, Theorem 3.1) *The following conditions are equivalent under the assumption that $\succsim$ is reflexive, transitive and satisfies Mixture Continuity of $\succsim$:*

(i) *for every $f \in \mathcal{F}$ and $\pi \in \Pi$, $f \sim \frac{1}{2} f + \frac{1}{2}\pi f$,*
(ii) *for every $f \in \mathcal{F}$, $\pi \in \Pi$ and $\alpha \in [0, 1]$, $f \sim \alpha\pi f + (1 - \alpha) f$,*
(iii) *for every $f \in \mathcal{F}$ and $\pi_i \in \Pi$, $f \sim \frac{1}{n} \sum_{i=1}^n \pi_i f$,*
(iv) *for every $f \in \mathcal{F}$, $\pi_i \in \Pi$ and $\alpha_i \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$, $f \sim \sum_{i=1}^n \alpha_i \pi_i f$, and*
(v) *for every $f \in \mathcal{F}$ and $\pi \in \Pi$, $f \sim^* \pi f$.*

Moreover, the above are equivalent to each of the following under Axioms 1–7:

(vi) *Event Symmetry,*
(vii) *for every $f, g \in \mathcal{F}$, if $\int f dp \succsim \int g dp$ for all symmetric $p \in \Delta(S^\infty)$, then $f \succsim g$.*

Finally, if, in addition, there exists an extension of $\succsim$ to $\hat{\mathcal{F}}$ that is reflexive, transitive and satisfies Norm Continuity, then the following is equivalent to all of the above:

(viii) *for* $f, g \in \mathcal{F}$, $f \succsim g$ *if and only if* $f^{\Psi} \hat{\succsim} g^{\Psi}$, *if* $\hat{\succsim}$ *is any such extension.*

A.2. Proof of Theorem 2

We begin by showing that the axioms imply the desired representation. Observing that Mixture Continuity of $\succsim$ implies Ghirardato et al.'s (2004) Archimedean axiom, we see that $\succsim$ are Invariant Biseparable preferences (i.e., satisfy axioms 1-5 in Ghirardato et al., 2004). By Proposition 7 in Ghirardato et al. (2004), $\succsim$ has a representation, $I(u(f))$, where u is non-constant and affine, and I is monotonic, constant linear and lies between $\min_{p\in C}\int u(f)\,dp$ and $\max_{p\in C}\int u(f)\,dp$ (i.e., for all simple acts f, $\min_{p\in C}\int u(f)\,dp \le I(u(f)) \le \max_{p\in C}\int u(f)\,dp$) where C is the Bewley set from Theorem 4.5 in Klibanoff et al. (2014). By that Theorem 4.5, $\min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp \le \min_{p\in C}\int u(f)\,dp$ and $\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp \ge \max_{p\in C}\int u(f)\,dp$. Therefore

$$\min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp \le I(u(f)) \le \max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp. \tag{9}$$

Now consider the Relevant Range axiom. Observe that $C^*(f)$ can be written as

$$\{x\in X: \min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp \le u(x) \le \max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp\}.$$

Thus, $C^*(f)=C^*(g)$ if

$$\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp = \max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(g)\,dp,$$

and

$$\min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp = \min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(g)\,dp.$$

Relevant Range therefore implies that $I(u(f))$ must be able to be expressed as a function of $\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp$ and $\min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp$ only. Since (9) holds and $I(u(f))$ depends only on $\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp$ and $\min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp$, we may apply Lemma B.5 in Ghirardato et al. (2004) to conclude that

$$I(u(f)) = \alpha \min_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp + (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}}\int u(f)\,dp,$$

for some $\alpha\in[0,1]$.

We next show that $R(\succsim)$ is finite. Consider $\alpha\in[0,1)$ first. Suppose $R(\succsim)$ is not finite. Then, we can take distinct $\ell_n\in R(\succsim)$ for each n. Let $A_n=\bigcup_{k>n}\Psi^{-1}(\ell_k)$. Then, $A_n\searrow\emptyset$. Without loss of generality, assume $[0,1]\subseteq u(X)$. Let $u(x)=1>u(x')=\frac{1}{k}>u(x'')=0$ for each integer $k>1$. By Monotone Continuity of $\succsim^*$, for each integer $k>1$, there is $n(k)>0$ such that $V(xA_{n(k)}x'')<\frac{1}{k}$. Since $V(xA_{n(k)}x'')$ is decreasing in $n(k)$, $V(xA_{n(k)}x'')\to 0$. However, applying the α-MEU form of V shows

$$V(xA_{n(k)}x'') \ge (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}} p(A_{n(k)}) = 1-\alpha > 0,$$

a contradiction. Note that the equality in the previous sentence follows since, no matter the value of $n(k)$, if $R(\succsim)$ is infinite there is an $m > n(k)$ with $\ell_m \in R(\succsim)$, which, by definition of A_n, ensures $\Psi^{-1}(\ell_m) \subseteq A_{n(k)}$ and therefore $\max_{p\in\{\ell^\infty:\ell\in R(\succsim)\}} p\left(A_{n(k)}\right) \geq \ell_m^\infty(A_{n(k)}) \geq \ell_m^\infty(\Psi^{-1}(\ell_m)) = 1$.

Now let $\alpha = 1$. Take $\ell_n \in R(\succsim)$ and $A_n \subset S^\infty$ as above, and also let $u(x) = 1 > u\left(x'\right) = \frac{1}{k} > u\left(x''\right) = 0$. By Monotone Continuity of $\succsim^*$, for each $k > 1$, there is $n(k) > 0$ such that $x' \succsim^* x A_{n(k)} x''$. Again invoking Theorem 4.5 from Klibanoff et al. (2014), $\frac{1}{k} \geq p\left(A_{n(k)}\right)$ for all $p \in C$. Equivalently, $1 - \frac{1}{k} \leq p\left(S^\infty \backslash A_{n(k)}\right)$ for all $p \in C$. This implies $V\left(x S^\infty \backslash A_{n(k)} x''\right) \in \left[1 - \frac{1}{k}, 1\right]$. Since $V\left(x S^\infty \backslash A_{n(k)} x''\right)$ is increasing in $n(k)$, $V\left(x S^\infty \backslash A_{n(k)} x''\right) \to 1$. However, $(\ell_{n+1})^\infty\left(S^\infty \backslash A_n\right) = 0$ for all n, and hence $V\left(x S^\infty \backslash A_{n(k)} x''\right) \to 0$, a contradiction.

This proves that $\succsim$ has the desired representation.

Next, we show the axioms are necessary for the representation in (6), and thus also (5). That $R(\hat{\succsim}) = D$ follows from Theorem 4.1 in Klibanoff et al. (2014). That Relevant Range is satisfied then follows since $C^*(f) = \{x \in X : \max_{p\in\left\{\ell^\infty:\ell\in R(\hat{\succsim})\right\}} \int u(f)\, dp \geq u(x) \geq \min_{p\in\left\{\ell^\infty:\ell\in R(\hat{\succsim})\right\}} \int u(f)\, dp\}$. The remaining axioms except Monotone Continuity of $\succsim^*$ are straightforward to verify.

We establish necessity of Monotone Continuity of $\succsim^*$. Consider $V_1(f) \equiv \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\, dp$ first. The Bewley set of V_1 is $co\left(\{\ell^\infty : \ell \in D\}\right)$ and it is weak* compact since D is finite. Thus, V_1 satisfies Monotone Continuity of $\succsim^*$. Similarly, $V_0(f) = \max_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\, dp$ also satisfies Monotone Continuity of $\succsim^*$. Take $A_n \searrow \emptyset$ and $x, x', x'' \in X$ such that $u\left(x'\right) > u\left(x''\right)$. Then, there are $\bar{n}_1$ and $\bar{n}_0$ such that

$$V_1\left(\lambda x' + (1-\lambda) h\right) \geq V_1\left(\lambda x A_n x'' + (1-\lambda) h\right),$$

for all $\lambda \in [0, 1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_1$, and

$$V_0\left(\lambda x' + (1-\lambda) h\right) \geq V_0\left(\lambda x A_n x'' + (1-\lambda) h\right),$$

for all $\lambda \in [0, 1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_2$. Since $V = \alpha V_1 + (1-\alpha) V_0$,

$$V\left(\lambda x' + (1-\lambda) h\right) \geq V\left(\lambda x A_n x'' + (1-\lambda) h\right) \text{ for } n = \max\left(\bar{n}_1, \bar{n}_2\right).$$

Thus, Monotone Continuity of $\succsim^*$ is satisfied.

We now show uniqueness. Uniqueness of D follows from uniqueness of $R\left(\succsim\right)$. Uniqueness of α when D is non-singleton is a conclusion of Lemma B.5 in Ghirardato et al. (2004). Uniqueness of u up to positive affine transformations is standard.

A.3. *Proof of Proposition 1*

The proof makes use of the following result establishing the overlap between SEU and preferences having a general α-MEU representation (1) with a finite set of measures C for a measurable space Ω without assuming $\Omega = S^\infty$ or symmetry.

Lemma 1. *Consider a preference relation represented by (1) and assume $C = \{p_1, ..., p_K\}$ with $K \geq 2$. The preference satisfies Independence if and only if $\alpha = \frac{1}{2}$ and there is $\widehat{p} \in \Delta\Omega$ such that $p \in C$ implies $2\widehat{p} - p \in co(C)$.*

We first prove the Proposition and then give the (somewhat lengthy) proof of the Lemma in its own subsection.

The "if" direction of the Proposition is implied by Lemma 1 with $\widehat{p} = \frac{1}{2}\ell_1^\infty + \frac{1}{2}\ell_2^\infty$ for $D = \{\ell_1, \ell_2\}$. For the "only if" direction, suppose the preference satisfies Independence. Then, there is $\widehat{p} \in \Delta(\Omega)$ such that $V(f) = \int u(f)\, d\widehat{p}$ for all $f \in \mathcal{F}$. By Event Symmetry, $\widehat{p}$ is a symmetric measure. By the de Finetti Theorem, $\widehat{p}$ is a mixture of i.i.d. measures. Moreover, for any $\ell \in D$, since D contains at least two elements, we have $V\left(1_{\Psi^{-1}(\ell)}\right) = 1 - \alpha = \widehat{p}\left(\Psi^{-1}(\ell)\right)$. Because $\alpha = \frac{1}{2}$ by Lemma 1,

$$1 = \widehat{p}(\Omega) \geq \sum_{\ell \in D} \widehat{p}\left(\Psi^{-1}(\ell)\right) = \sum_{\ell \in D} \frac{1}{2} \geq 1.$$

The latter inequality holds because D has at least two elements. To maintain equality, conclude there must be exactly two elements in D.

A.3.1. Proof of Lemma 1

Consider the "if" direction first. Assume the properties and take any $f \in \mathcal{F}$. Without loss of generality, assume $\int u(f)\, dp_1 = \min_{p \in C} \int u(f)\, dp$ and $\int u(f)\, dp_2 = \max_{p \in C} \int u(f)\, dp$. Then, $2\widehat{p} - p_1$ and $2\widehat{p} - p_2$ belong to $co(C)$. Hence,

$$\begin{aligned}
V(f) &= \frac{1}{2} \min_{p \in C} \int u(f)\, dp + \frac{1}{2} \max_{p \in C} \int u(f)\, dp \\
&= \frac{1}{2} \int u(f)\, dp_1 + \frac{1}{2} \max_{p \in C} \int u(f)\, dp \\
&\geq \frac{1}{2} \int u(f)\, dp_1 + \frac{1}{2} \int u(f)\, d(2\widehat{p} - p_1) = \int u(f)\, d\widehat{p}
\end{aligned}$$

and

$$\begin{aligned}
V(f) &= \frac{1}{2} \min_{p \in C} \int u(f)\, dp + \frac{1}{2} \max_{p \in C} \int u(f)\, dp \\
&= \frac{1}{2} \min_{p \in C} \int u(f)\, dp + \frac{1}{2} \int u(f)\, dp_2 \\
&\leq \frac{1}{2} \int u(f)\, d(2\widehat{p} - p_2) + \frac{1}{2} \int u(f)\, dp_2 = \int u(f)\, d\widehat{p}
\end{aligned}$$

This implies $V(f) = \int u(f)\, d\widehat{p}$. This holds for any f and thus the preference satisfies Independence.

Now consider the "only if" direction. Suppose the preference satisfies Independence. Then

$$V(\lambda f + (1 - \lambda) g) = \lambda V(f) + (1 - \lambda) V(g)$$

for each $f, g \in \mathcal{F}$ and $\lambda \in [0, 1]$.

We show that there are $f, g \in \mathcal{F}$ and $x \in X$ such that $\min_{p \in C} \int u(f)\, dp \neq \max_{p \in C} \int u(f)\, dp$ and $\frac{1}{2} f(\omega) + \frac{1}{2} g(\omega) \sim x$ for all ω. Because u is non-constant, we can let $u(X) \supset [-1, 1]$ and we do so. Then, there is an $\eta \in B_0(\Omega, \mathbb{R})$ such that $\min_{p \in C} \int \eta dp \neq \max_{p \in C} \int \eta dp$ because C has multiple elements. By normalization, we can assume

$$-1 \leq \inf_{\omega \in \Omega} \eta(\omega) < 0 < \sup_{\omega \in \Omega} \eta(\omega) \leq 1.$$

Then, there are $f, g \in \mathcal{F}$ such that $\eta = u \circ f$ and $-\eta = u \circ g$. Letting $x \in X$ such that $u(x) = 0$, we see that $\frac{1}{2} f(\omega) + \frac{1}{2} g(\omega) \sim x$ for all ω.

Then, for the above $f, g \in \mathcal{F}$ and $x \in X$,

$$\begin{aligned}
0 = u(x) =& V\left(\frac{1}{2} f + \frac{1}{2} g\right) = \frac{1}{2} V(f) + \frac{1}{2} V(g) \\
=& \frac{1}{2}\left(\alpha \min_{p \in C} \int \eta dp + (1-\alpha) \max_{p \in C} \int \eta dp\right) \\
&+ \frac{1}{2}\left(\alpha \min_{p \in C} \int -\eta dp + (1-\alpha) \max_{p \in C} \int -\eta dp\right) \\
=& \frac{1}{2}(2\alpha - 1)\left(\min_{p \in C} \int u(f) dp - \max_{p \in C} \int u(f) dp\right),
\end{aligned}$$

where the third equality follows from Independence of the preference. This implies

$$(2\alpha - 1)\left(\min_{p \in C} \int u(f) dp - \max_{p \in C} \int u(f) dp\right) = 0.$$

Because $\min_{p \in C} \int u(f) dp \neq \max_{p \in C} \int u(f) dp$, $\alpha = \frac{1}{2}$.

Turn to the property that there is $\widehat{p} \in \Delta\Omega$ such that $p \in C$ implies $2\widehat{p} - p \in co(C)$. Note that Independence implies V is SEU and we can find $\widehat{p} \in \Delta\Omega$ such that $V(f) = \int u(f) d\widehat{p}$ for all $f \in \mathcal{F}$. Without loss of generality, because $p_1 \in C$, suppose $2\widehat{p} - p_1 \notin C$ by contradiction. A separating hyperplane theorem (for example, see the references in Ghirardato and Siniscalchi, 2012, Footnote 14) implies that there is $f \in \mathcal{F}$ such that $\int u(f) d(2\widehat{p} - p_1) > \max_{p \in C} \int u(f) dp$. But then,

$$\begin{aligned}
V(f) &= \frac{1}{2} \min_{p \in C} \int u(f) dp + \frac{1}{2} \max_{p \in C} \int u(f) dp \\
&\leq \frac{1}{2} \int u(f) dp_1 + \frac{1}{2} \max_{p \in C} \int u(f) dp \\
&< \frac{1}{2} \int u(f) dp_1 + \frac{1}{2} \int u(f) d(2\widehat{p} - p_1) = \int u(f) d\widehat{p}.
\end{aligned}$$

This contradicts the property $V(f) = \int u(f) d\widehat{p}$ for all $f \in \mathcal{F}$. Conclude that $p \in C$ implies $2\widehat{p} - p \in co(C)$.

A.4. Proof that i.i.d. α-MEU preferences satisfy Kopylov's (2003) axioms

Kopylov (2003, Theorem 2.4) shows that a set of axioms is equivalent to preferences being represented by a functional of the form

$$U_K(f) = \alpha_0 \min_{m \in \mathcal{M}_0} \int u(f) dm + (1 - \alpha_0) \max_{m \in \mathcal{M}_0} \int u(f) dm. \tag{10}$$

Here, $\alpha_0 \in [0, 1]$ is a constant, and $\mathcal{M}_0 \subset \Delta(\Omega)$ is the set of probability measures m such that

$$U_0(r) = \int u(r) dm$$

represents the preference restricted to $\mathcal{G}_0$, the set of subjectively risky acts. An act r is a subjectively risky act, i.e., $r \in \mathcal{G}_0$ if and only if for all acts f, g and all $\lambda \in (0, 1)$,

$$f \succsim g \Leftrightarrow \lambda f + (1-\lambda) r \succsim \lambda g + (1-\lambda) r.$$

Theorem 6. *All i.i.d. α-MEU preferences as in (5) or (6) have a representation as in (10), and satisfy Kopylov's axioms.*

Proof of Theorem 6. Most of the argument proceeds assuming Ω is any measurable space and that the set of measures in the α-MEU representation is finite. The further restrictions to $\Omega = S^\infty$ and the measures being i.i.d. are imposed only in the last step.

Suppose that preferences are represented by

$$V(f) \equiv \alpha \min_{p \in P} \int u(f)\, dp + (1-\alpha) \max_{p \in P} \int u(f)\, dp \tag{11}$$

where $P = \{p_1, ..., p_K\}$ and $\alpha \neq \frac{1}{2}$. The next two lemmata find the sets $\mathcal{G}_0$ and $\mathcal{M}_0$ for such preferences. The case of $\alpha = \frac{1}{2}$ will be dealt with separately.

Lemma 2. *For such preferences, $\mathcal{G}_0 = \left\{ r \in \mathcal{F} : \min_{p \in P} \int u(r)\, dp = \max_{p \in P} \int u(r)\, dp \right\}$*

Proof of Lemma 2. First we show $\supset$. Suppose $\min_{p \in P} \int u(r)\, dp = \max_{p \in P} \int u(r)\, dp$. Then, for any $f \in \mathcal{F}$ and $\lambda \in [0,1]$,

$$\min_{p \in P} \int u(\lambda f + (1-\lambda) r)\, dp = \lambda \min_{p \in P} \int u(f)\, dp + (1-\lambda) \min_{p \in P} \int u(r)\, dp$$

and similarly for the maximum operator. Thus, $V(\lambda f + (1-\lambda) r) = \lambda V(f) + (1-\lambda) V(r)$ and hence $r \in \mathcal{G}_0$.

Turn to $\subset$. Suppose $r \in \mathcal{G}_0$. Take $x \in X$ such that

$$u(x) = \frac{1}{2} \left(\max_{\omega \in \Omega} u(r(\omega)) + \min_{\omega \in \Omega} u(r(\omega)) \right),$$

where the max and min exist because all acts are simple acts. Since $u(X)$ is convex, such an x exists. Take an $f \in \mathcal{F}$ such that $u(f(\omega)) = 2u(x) - u(r(\omega))$. To see such an f exists, note that $u(X)$ is convex and

$$\min_{\omega \in \Omega} u(r(\omega)) \leq 2u(x) - u(r(\omega)) \leq \max_{\omega \in \Omega} u(r(\omega))$$

implies $u(f(\omega)) = 2u(x) - u(r(\omega)) \in u(X)$. We see that $\frac{1}{2} f(\omega) + \frac{1}{2} r(\omega) \sim x$ for all $\omega \in \Omega$. Take $x' \in X$ such that $f \sim x'$. Then,

$$\begin{aligned}
u(x) =& V\left(\frac{1}{2} f + \frac{1}{2} r\right) = V\left(\frac{1}{2} x' + \frac{1}{2} r\right) = \frac{1}{2} V\left(x'\right) + \frac{1}{2} V(r) = \frac{1}{2} V(f) + \frac{1}{2} V(r) \\
=& \frac{1}{2} \left(\alpha \min_{p \in P} \int u(f)\, dp + (1-\alpha) \max_{p \in P} \int u(f)\, dp \right) \\
&+ \frac{1}{2} \left(\alpha \min_{p \in P} \int u(r)\, dp + (1-\alpha) \max_{p \in P} \int u(r)\, dp \right) \\
=& \frac{1}{2} \left(\alpha \min_{p \in P} \left(2u(x) - \int u(r)\, dp \right) + (1-\alpha) \max_{p \in P} \left(2u(x) - \int u(r)\, dp \right) \right) \\
&+ \frac{1}{2} \left(\alpha \min_{p \in P} \int u(r)\, dp + (1-\alpha) \max_{p \in P} \int u(r)\, dp \right)
\end{aligned}$$

$$=\frac{1}{2}\left(2u\left(x\right)+\left(2\alpha-1\right)\left(\min_{p\in P}\int u\left(r\right)dp-\max_{p\in P}\int u\left(r\right)dp\right)\right).$$

Here, the second equality follows because $r\in\mathcal{G}_0$ and $f\sim x'$. The third holds because x' is a lottery. Then, the above computation results in

$$0=\left(2\alpha-1\right)\left(\min_{p\in P}\int u\left(r\right)dp-\max_{p\in P}\int u\left(r\right)dp\right).$$

Because $\alpha\neq\frac{1}{2}$, conclude $\min_{p\in P}\int u\left(r\right)dp=\max_{p\in P}\int u\left(r\right)dp$. $\square$

Lemma 3. *For such preferences,*

$$\mathcal{M}_0=\left\{p\in\Delta(\Omega):p=\sum_{k=1}^{K}\theta_k p_k \textit{ for } \sum_{k=1}^{K}\theta_k=1 \textit{ and } \theta_1,\ldots,\theta_K\in\mathbb{R}\right\}$$

Notice that the weights $\theta_1,\ldots,\theta_K$ need not all be nonnegative.

Proof of Lemma 3. First, suppose $p=\sum_{k=1}^{K}\theta_k p_k\in\Delta\Omega$ for some $\theta_1,\ldots,\theta_K\in\mathbb{R}$ with $\sum_{k=1}^{K}\theta_k=1$. Then, for any $r\in\mathcal{G}_0$,

$$\int u\left(r\right)dp=\sum_{k=1}^{K}\theta_k\int u\left(r\right)dp_k=\int u\left(r\right)dp_1=V\left(r\right)$$

by Lemma 2. Thus, SEU with p as the measure evaluates all $r\in\mathcal{G}_0$ correctly, and therefore $p\in\mathcal{M}_0$.

Turn to the opposite direction. Suppose $p\in\Delta(\Omega)$ such that, for all $\theta_1,\ldots,\theta_K\in\mathbb{R}$ satisfying $\sum_{k=1}^{K}\theta_k=1$, $p\neq\sum_{k=1}^{K}\theta_k p_k$. Let

$$C=\left\{p'\in ba\left(\Omega\right):p'=\sum_{k=1}^{K}\theta_k p_k \text{ for some } \theta_1,\ldots,\theta_K\in\mathbb{R} \text{ with } \sum_{k=1}^{K}\theta_k=1\right\}.$$

Then C is closed and convex, and by the separating hyperplane theorem there is $b\in B\left(\Delta\left(\Omega\right),\mathbb{R}\right)$ such that

$$\int bdp>\int bd\left(\sum_{k=1}^{K}\theta_k p_k\right) \tag{12}$$

for all $\theta_1,\ldots,\theta_K\in\mathbb{R}$ satisfying $\sum_{k=1}^{K}\theta_k=1$. In fact, we can take $b\in B_0\left(\Delta\left(\Omega\right),\mathbb{R}\right)$ by Ghirardato and Siniscalchi (2012, Footnote 14), and also $b\in B_0\left(\Delta\left(\Omega\right),u\left(X\right)\right)$ by normalization. Then, there is $r\in\mathcal{F}$ such that $b=u\circ r$. We now show that $\max_{k=1,\ldots,K}\int u\left(r\right)dp_k=\min_{k=1,\ldots,K}\int u\left(r\right)dp_k$. Suppose this does not hold. Without loss of generality, let $\int u\left(r\right)dp_1>\int u\left(r\right)dp_2$. Then, we can take a very large θ_1 and a very small θ_2, keeping $\theta_1+\theta_2,\theta_3,\ldots,\theta_K$ constant. This makes the right-hand side of (12) become as large as we like, which is a contradiction. Thus, $r\in\mathcal{G}_0$ by Lemma 2. But then $p\notin\mathcal{M}_0$, since $\int u\left(r\right)dp\neq\int u\left(r\right)dp_k$ for all $k=1,\ldots,K$, whereas the equality of these expectations is required for all acts in $\mathcal{G}_0$. $\square$

Lemma 4. *For such preferences, if $\Omega=S^{\infty}$ and each $p_k\in P$ is i.i.d., then $\mathcal{M}_0=co\left\{P\right\}$.*

Proof of Lemma 4. By Lemma 3,

$$\mathcal{M}_0 = \left\{ p \in \Delta(\Omega) : p = \sum_{k=1}^{K} \theta_k p_k \text{ for } \sum_{k=1}^{K} \theta_k = 1 \text{ and } \theta_1, ..., \theta_K \in \mathbb{R} \right\}.$$

If $K = 1$ the result is trivial. Assume $K \geq 2$. If $\theta_i < 0$ for some $i \in \{1, \ldots, K\}$, then $\sum_{k=1}^{K} \theta_k p_k$ is not a probability measure because, letting $\ell_i \in \Delta(S)$ be such that $p_i = \ell_i^\infty$,

$$p\left(\Psi^{-1}(\ell_i)\right) = \sum_{k=1}^{K} \theta_k p_k \left(\Psi^{-1}(\ell_i)\right) = \theta_i < 0.$$

Therefore

$$\mathcal{M}_0 = \left\{ p \in \Delta(\Omega) : p = \sum_{k=1}^{K} \theta_k p_k \text{ for } \sum_{k=1}^{K} \theta_k = 1 \text{ and } \theta_1, ..., \theta_K \in \mathbb{R}_+ \right\} = co\{P\},$$

which completes the proof. □

Thus far, we have shown that any i.i.d. α-MEU representation as in (5) or (6) with $\alpha \neq \frac{1}{2}$ has $\mathcal{M}_0 = co\{\{\ell^\infty : \ell \in D\}\} = co\left\{\{\ell^\infty : \ell \in R(\succsim)\}\right\}$ and therefore is also a representation of the form (10). By Kopylov 2003, Theorem 2.4, this implies that it satisfies his axioms.

Before turning to the case of $\alpha = \frac{1}{2}$, we note that the i.i.d. restriction in Lemma 4 was important to the argument. More general preferences represented as in (11) with $\alpha \neq \frac{1}{2}$ may not have representations of the form (10). For a simple example, suppose that $\Omega = S = \{H, T\}$ and $P = \{p_1, p_2\}$ with $p_1(H) = \frac{1}{3}$ and $p_2(H) = \frac{2}{3}$. Then $\mathcal{M}_0 = \Delta(S) \supset co\{P\}$ implying that no representation as in (10) exists, and that these preferences violate Kopylov's axiom of Partial Ignorance outside $\mathcal{G}_0$.

Finally, turn to i.i.d. α-MEU preferences (so that $\Omega = S^\infty$ and each $p_k \in P$ is i.i.d.) with $\alpha = \frac{1}{2}$. If $K \leq 2$, then, by Proposition 1, preferences are SEU and thus all acts are subjectively risky and Kopylov's axioms are satisfied. Suppose therefore that $K \geq 3$. Inspection of the proofs of Lemma 2, Lemma 3 and Lemma 4 reveals that the parts of those arguments that did not depend on $\alpha \neq \frac{1}{2}$ can be used to show that $\mathcal{M}_0 \subseteq co\{P\}$. We complete the argument that $\mathcal{M}_0 = co\{P\}$ by showing that $\mathcal{M}_0 \supseteq co\{P\}$. Suppose $r \in \mathcal{G}_0$. We show that $\min_{p \in P} \int u(r)\, dp = \max_{p \in P} \int u(r)\, dp$. Suppose not. Let $p_k = \ell_k^\infty$ with some $\ell_k \in \Delta(S)$ for $k = 1, ..., K$. Without loss of generality, assume $\max_{p \in P} \int u(r)\, dp = \int u(r)\, dp_1 = \int u(r)\, d\ell_1^\infty > \int u(r)\, d\ell_2^\infty = \int u(r)\, dp_2 = \min_{p \in P} \int u(r)\, dp$. Since $r \in \mathcal{G}_0$, for any f, g such that $f \succsim g$ and all $\lambda \in [0, 1]$ it must be that $V(\lambda f + (1-\lambda)r) \geq V(\lambda g + (1-\lambda)r)$. Let

$$f = r(\arg\max_{\omega \in \Omega} u(r(\omega)))\Psi^{-1}(\ell_1 \cup \ell_3) r(\arg\min_{\omega \in \Omega} u(r(\omega))) \text{ and}$$

$$g = r(\arg\max_{\omega \in \Omega} u(r(\omega)))\Psi^{-1}(\ell_2 \cup \ell_3) r(\arg\min_{\omega \in \Omega} u(r(\omega))).$$

Observe that $f \sim g$. Furthermore, for all λ,

$$\begin{aligned} &V(\lambda f + (1-\lambda)r) \\ =&\frac{1}{2}(\lambda \min_{\omega \in \Omega} u(r(\omega)) + (1-\lambda)\int u(r)\, d\ell_2^\infty) + \frac{1}{2}(\lambda \max_{\omega \in \Omega} u(r(\omega)) + (1-\lambda)\int u(r)\, d\ell_1^\infty) \end{aligned}$$

while, for λ sufficiently large,

$$
\begin{aligned}
&V(\lambda g+(1-\lambda)r)\\
=&\frac{1}{2}(\lambda\min_{\omega\in\Omega}u(r(\omega))+(1-\lambda)\int u(r)\,d\ell_1^\infty)+\frac{1}{2}(\lambda\max_{\omega\in\Omega}u(r(\omega))+(1-\lambda)\int u(r)\,d\ell_3^\infty).
\end{aligned}
$$

Therefore, $V(\lambda f+(1-\lambda)r)=V(\lambda g+(1-\lambda)r)$ if and only if $\int u(r)\,d\ell_3^\infty=\int u(r)\,d\ell_2^\infty$. Next, consider

$$
\begin{aligned}
f'&=r(\arg\min_{\omega\in\Omega}u(r(\omega)))\Psi^{-1}(\ell_1\cup\ell_3)r(\arg\max_{\omega\in\Omega}u(r(\omega)))\text{ and}\\
g'&=r(\arg\min_{\omega\in\Omega}u(r(\omega)))\Psi^{-1}(\ell_2\cup\ell_3)r(\arg\max_{\omega\in\Omega}u(r(\omega))).
\end{aligned}
$$

Again, $f'\sim g'$. Furthermore, for λ sufficiently large,

$$
\begin{aligned}
&V\left(\lambda f'+(1-\lambda)r\right)\\
=&\frac{1}{2}(\lambda\min_{\omega\in\Omega}u(r(\omega))+(1-\lambda)\int u(r)\,d\ell_3^\infty)+\frac{1}{2}(\lambda\max_{\omega\in\Omega}u(r(\omega))+(1-\lambda)\int u(r)\,d\ell_2^\infty)
\end{aligned}
$$

and, for all λ,

$$
\begin{aligned}
&V\left(\lambda g'+(1-\lambda)r\right)\\
=&\frac{1}{2}(\lambda\min_{\omega\in\Omega}u(r(\omega))+(1-\lambda)\int u(r)\,d\ell_2^\infty)+\frac{1}{2}(\lambda\max_{\omega\in\Omega}u(r(\omega))+(1-\lambda)\int u(r)\,d\ell_1^\infty).
\end{aligned}
$$

Therefore, $V\left(\lambda f'+(1-\lambda)r\right)=V\left(\lambda g'+(1-\lambda)r\right)$ if and only if $\int u(r)\,d\ell_3^\infty=\int u(r)\,d\ell_1^\infty$. Since $\int u(r)\,d\ell_1^\infty>\int u(r)\,d\ell_2^\infty$, it must be that, for λ sufficiently large, either $\lambda f+(1-\lambda)r\nsim\lambda g+(1-\lambda)r$ or $\lambda f'+(1-\lambda)r\nsim\lambda g'+(1-\lambda)r$ must hold, contradicting $r\in\mathcal{G}_0$. Therefore, $\mathcal{G}_0=\left\{r\in\mathcal{F}:\min_{p\in P}\int u(r)\,dp=\max_{p\in P}\int u(r)\,dp\right\}$ and therefore $co\{P\}\subseteq\mathcal{M}_0$. This completes the $\alpha=\frac{1}{2}$ case and the argument. □

A.5. Proof of Theorem 3

We start with necessity of Monotone Continuity of $\succsim^*$ and Savage's axioms on $\mathcal{F}^\Psi$. Ghirardato and Siniscalchi (2010) show necessity of Cauchy continuity. Necessity of the remaining axioms is straightforward.

Monotone Continuity of $\succsim^*$: Suppose that there are $m,M>0$ such that $m|a-b|\le|\phi(a)-\phi(b)|\le M|a-b|$ for all $a,b\in u(X)$. Fix any $x,x',x''\in X$ with $x'\succ x''$. The only non-trivial case is $x\succ x'$. Without loss of generality, assume $u(x)=1>u\left(x'\right)=t'>u\left(x''\right)=0$ and $[0,1]\subseteq u(X)$. Suppose $A_n\searrow\emptyset$. Take $\varepsilon',\varepsilon>0$ so that

$$\varepsilon'<t'\text{ and }m\left(t'-\varepsilon'\right)(1-\varepsilon)\ge M\left(1-t'\right)\varepsilon.$$

Define $\zeta_n:\Delta(S)\to\mathbb{R}$ by $\zeta_n(\ell)=\ell^\infty(A_n)$, and temporarily equip $\Delta(S)$ with the wc topology. Since wc open sets are weak* open, μ is well-defined on the Borel σ-algebra generated by wc open sets. Then, by Lusin's theorem (Aliprantis and Border, 2006, Theorem 12.8), there is a wc compact set $L\subseteq\Delta(S)$ such that $\mu(L)>1-\varepsilon$ and all ζ_n are wc continuous. Note that ζ_n converges monotonically to 0 pointwise. Then by Dini's Theorem (Aliprantis and Border, 2006, Theorem 2.66), ζ_n on L converges uniformly to 0. Hence there is $N>0$ such that $\zeta_N=\ell^\infty(A_N)<\varepsilon'$ for all $\ell\in L$. To see $x'\succsim^* xA_Nx''$, and thus Monotone Continuity of $\succsim^*$, compute, for any $\alpha\in[0,1]$ and $h\in\mathcal{F}$,

$$
\begin{aligned}
&U\left(\alpha x' + (1-\alpha) h\right) - U\left(\alpha x A_N x'' + (1-\alpha) h\right) \\
&= \int_L \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha \ell^\infty (A_N) + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
&+ \int_{\Delta(S)\setminus L} \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha \ell^\infty (A_N) + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
&> \int_L \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha \varepsilon' + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
&+ \int_{\Delta(S)\setminus L} \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
&\geq \int_L \alpha m\left(t' - \varepsilon'\right) d\mu(\ell) + \int_{\Delta(S)\setminus L} \alpha M\left(t' - 1\right) d\mu(\ell) \\
&= \alpha\left[m\left(t' - \varepsilon'\right)\mu(L) - M\left(1 - t'\right)(1 - \mu(L))\right] \\
&\geq \alpha\left[m\left(t' - \varepsilon'\right)(1 - \varepsilon) - M\left(1 - t'\right)\varepsilon\right] \geq 0.
\end{aligned}
$$

P1-P6 on $\mathcal{F}^\Psi$: For $f \in \mathcal{F}^\Psi$, f is constant on $\Psi^{-1}(\ell)$, so

$$
\begin{aligned}
U(f) &= \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)\, d\ell^\infty\right) d\mu(\ell) \\
&= \int_{\Delta(S)} \phi\left(u\left(f \circ \Psi^{-1}(\ell)\right)\right) d\mu(\ell),
\end{aligned}
$$

represents $\succsim$ on $\mathcal{F}^\Psi$. Viewing $f \circ \Psi^{-1}(\ell)$ as an act from $\Delta(S)$ to X, this is an expected utility representation with countably additive, non-atomic μ and non-constant vNM utility function $v \equiv \phi \circ u$. Therefore, P1-P6 are satisfied. That the continuity axiom is satisfied follows by Lebesgue's dominated convergence theorem (Aliprantis and Border, 2006, Theorem 11.21).

As for sufficiency, we first prove the following claims.

Claim I. *There exists $u : X \to \mathbb{R}$ non-constant and affine that represents $\succsim$ on X. Moreover, without loss of generality* $[0, 1] \subseteq u(X)$.

This claim follows by standard results, see for example Cerreia-Vioglio et al. (2011).

Now for $f \in \mathcal{F}^\Psi$, let $u(f \circ \Psi^{-1}) : \Delta(S) \to \mathbb{R}$ denote the mapping defined by $u(f \circ \Psi^{-1})(\ell) = u(f(\Psi^{-1}(\ell)))$ for every $\ell \in \Delta(S)$. Note that $u(f \circ \Psi^{-1})$ is well-defined and belongs to $B_0(\Delta(S), u(X))$ since f is constant on $\Psi^{-1}(\ell)$.

Claim II. $B_0(\Delta(S), u(X)) = \{u(f \circ \Psi^{-1}) : f \in \mathcal{F}^\Psi\}$.

Showing that $\{u(f \circ \Psi^{-1}) : f \in \mathcal{F}^\Psi\} \subseteq B_0(\Delta(S), u(X))$ is straightforward. As for $B_0(\Delta(S), u(X)) \subseteq \{u(f \circ \Psi^{-1}) : f \in \mathcal{F}^\Psi\}$, take $a \in B_0(\Delta(S), u(X))$. Then $a = \sum_{i=1}^n y_i 1_{A_i}$ where $(A_i)_{i=1}^n$ is a measurable partition of $\Delta(S)$. For each i, let $E_i = \Psi^{-1}(A_i)$. Note that if A_i is non-empty, then E_i is also non-empty. Now take any $x_1, \ldots, x_n \in X$ such that $u(x_i) = y_i$ for every $i = 1, \ldots, n$. Let $f \in \mathcal{F}^\Psi$ be such that for every $i = 1, \ldots n$ it holds that $f(\omega) = x_i$ whenever $\omega \in E_i$. Then $u(f \circ \Psi^{-1}) = a$. It follows that $B_0(\Delta(S), u(X)) \subseteq \{u(f \circ \Psi^{-1}) : f \in \mathcal{F}^\Psi\}$ as desired.

Now define $\tilde{\succsim}$ on $B_0(\Delta(S), u(X))$ by

$$a \tilde{\succsim} b \iff \exists f, g \in \mathcal{F}^\Psi \text{ such that } u(f \circ \Psi^{-1}) = a, u(g \circ \Psi^{-1}) = b \text{ and } f \succsim g,$$

for every $a, b \in B_0(\Delta(S), u(X))$. Note that $\tilde{\succsim}$ is well defined since by Claim II for every $a, b \in B_0(\Delta(S), u(X))$ there exists $f, g \in \mathcal{F}^\Psi$ such that $a = u(f \circ \Psi^{-1})$ and $b = u(f \circ \Psi^{-1})$. Moreover, if $f, h \in \mathcal{F}^\Psi$ are such that $u(f \circ \Psi^{-1}) = u(h \circ \Psi^{-1})$ then $f \sim h$. To see this, observe that this last equality implies that $u(f(\omega)) = u(h(\omega))$ for every $\omega \in \{\omega : \Psi(\omega) \text{ is defined}\}$. Because Event Symmetry implies that the event $\{\omega : \Psi(\omega) \text{ is not defined}\}$ is null, it follows that $f \sim h$. Hence $\tilde{\succsim}$ is well defined. It is now straightforward to see that $\tilde{\succsim}$ satisfies the following claim:

Claim III. $\tilde{\succsim}$ *satisfies:*

P1 *$\tilde{\succsim}$ is complete and transitive.*

P2 *For every $a, b, c, c' \in B_0(\Delta(S), u(X))$ and $E \in \Sigma^\Delta$,*

$$aEc \tilde{\succsim} bEc \implies aEc' \tilde{\succsim} aEc'.$$

P3 *For every non-null $A \in \Sigma^\Delta$, $x, y \in u(X)$ and $a, b \in B_0(\Delta(S), u(X))$,*

$$x > y \iff xAa \tilde{\succ} yAb.$$

P4 *For every $A, B \in \Sigma^\Delta$ and $x, y, x', y' \in u(X)$ such that $x \tilde{\succ} y$, $x' \tilde{\succ} y'$,*

$$xAy \tilde{\succsim} xBy \implies x'Ay' \tilde{\succsim} x'By'.$$

P6 *For every $a, b \in B_0(\Delta(S), u(X))$ and $x \in u(X)$ such that $b \tilde{\succ} a$, there exists a Σ^Δ-measurable finite partition $(A_i)_{i=1}^n$ of $\Delta(S)$ such that for every $i = 1, \ldots, n$, $b \tilde{\succ} x_{A_i} a$ and $x_{A_i} b \tilde{\succ} a$.*

Recall that we write $a_n \to a$ if for every $b \in B_0(\Delta(S), u(X))$, $b \tilde{\succ} a$ implies that there exists N such that $n \geq N \implies b \tilde{\succ} a_n$ and $a \tilde{\succ} b$ implies that there exists N' such that $n \geq N' \implies a_n \tilde{\succ} b$.

Continuity. If (a_n) is a sequence in $B_0(\Delta(S), u(X))$, converges to $a \in B$ pointwise and satisfies $m \leq a_n \leq M$ for $m, M \in \mathbb{R}$, then $a_n \to a$.

Note that the σ-algebra generated by the open sets of $\Delta(S)$ is countably separated Mackey (1957) since S is a compact metric space. By Claim III and Stanca (2020, Theorem 5) (reported as Theorem 7 below), there exists $\mu \in \Delta(\Delta(S))$ non-atomic and $\phi : u(X) \to \mathbb{R}$ continuous and strictly increasing such that

$$a \tilde{\succsim} b \iff \int \phi(a(\ell))\mu(\ell) \geq \int \phi(b(\ell))\mu(\ell).$$

Moreover, if (μ', ϕ') is another representation then $\mu = \mu'$ and $\phi' = a\phi + b$ for constants a, b with $a > 0$. We can conclude that for every $f, g \in \mathcal{F}^\Psi$,

$$\begin{aligned} f \succsim g &\iff u(f \circ \Psi^{-1}) \tilde{\succsim} u(g \circ \Psi^{-1}) \\ &\iff \int \phi(u(f(\Psi^{-1}(\ell))))d\mu(\ell) \geq \int \phi(u(g(\Psi^{-1}(\ell))))d\mu(\ell) \\ &\iff \int \phi\Big(\int u(f(\omega))d\ell^\infty(\omega)\Big)d\mu(\ell) \geq \int \phi\Big(\int u(g(\omega))d\ell^\infty(\omega)\Big)d\mu(\ell). \end{aligned}$$

Theorem 7. (Stanca, 2020, Theorem 5) *Consider a measurable space $(\Xi, \mathcal{A})$ where $\mathcal{A}$ is countably separated. Let $\tilde{\succsim}$ be a relation on $B_0(\Xi, K)$ where K is an interval that contains 0 and a positive number. Then $\tilde{\succsim}$ satisfies P1-P6 and continuity if and only if there exists a non-atomic measure μ on $\Delta(\Xi)$ and $u : K \to \mathbb{R}$ strictly increasing and continuous such that*

$$U(f) = \int u(f(\xi))\mu(d\xi),$$

represents $\tilde{\succsim}$. Moreover, any other such representation with μ' and u' satisfies $\mu = \mu'$ and $u = au + b$ for $a > 0$ and constant b.

It remains to extend the representation to the entire set $\mathcal{F}$. By Proposition 1 in Ghirardato and Siniscalchi (2010), Cauchy continuity ensures the existence of a complete, monotonic and norm-continuous extension of $\succsim$ from $\mathcal{F}$ to $\hat{\mathcal{F}}$.[16] Denote this extension by $\hat{\succsim}$. When restricted to the set of bounded Σ^{Ψ}-measurable functions from Ω to X, it is represented by

$$V(f) = \int\limits_{\Delta(S)} \phi\left(\int\limits_{S^\infty} u(f)\, d\ell^\infty\right) d\mu(\ell).$$

Given the extension $\hat{\succsim}$, we can invoke the equivalence of (vi) and (viii) in Theorem 5. By this equivalence, for any act $f \in \mathcal{F}$, $f \sim x \in X$ if and only if $f^{\Psi} \hat{\sim} x^{\Psi} = x$ (f^{Ψ} is defined just before the statement of Theorem 5). Therefore, for any act $f \in \mathcal{F}$, $f \hat{\sim} f^{\Psi}$. Defining $U(f)$ by $U(f) = V(f^{\Psi})$, we see that U represents $\succsim$ on $\mathcal{F}$. Since, by construction of f^{Ψ}, $\int_{S^\infty} u(f^{\Psi})\, d\ell^\infty = \int_{S^\infty} u(f)\, d\ell^\infty$ for all $\ell \in \Delta(S)$,

$$\begin{aligned} U(f) = V(f^{\Psi}) &= \int\limits_{\Delta(S)} \phi\left(\int\limits_{S^\infty} u(f^{\Psi})\, d\ell^\infty\right) d\mu(\ell) \\ &= \int\limits_{\Delta(S)} \phi\left(\int\limits_{S^\infty} u(f)\, d\ell^\infty\right) d\mu(\ell) \ \text{ for } f \in \mathcal{F}. \end{aligned}$$

Finally, we need to show there are $m, M > 0$ such that $m|a - b| \le |\phi(a) - \phi(b)| \le M|a - b|$ for all $a, b \in u(X)$. To prove it by contradiction, assume that this does not hold. Because μ is non-atomic, $\operatorname{supp}\mu$ is infinite and thus we can take distinct elements $\ell_n \in \operatorname{supp}\mu$, for all integers $n \ge 1$. Let $A_n = \bigcup_{k \ge n} \Psi^{-1}(\ell_k)$. Then, $A_n \searrow \emptyset$ and for each n, $(\ell_n)^\infty(A_n) = 1 > \varepsilon \equiv \frac{1}{2}$. Fix $x \succ x' \succ x''$ and without loss of generality, assume $u(x) = 1 > u(x') = \frac{\varepsilon}{2} > u(x'') = 0$. Since $A_n \searrow \emptyset$, Monotone Continuity of $\succsim^*$ implies that there is n such that

$$\begin{aligned} &\int \phi\left(\alpha \tfrac{\varepsilon}{2} + (1 - \alpha) \int u(h)\, d\ell^\infty\right) d\mu(\ell) \\ &\ge \int \phi\left(\alpha \ell^\infty(A_n) + (1 - \alpha) \int u(h)\, d\ell^\infty\right) d\mu(\ell), \end{aligned} \tag{13}$$

[16] Recall that $\hat{\mathcal{F}}$ is the set of all bounded acts, i.e. measurable functions $f : \Omega \to X$ such that for some $x, y \in X$ it holds that $x \succsim f(\omega) \succsim y$ for every $\omega \in \Omega$.

for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$. To pick a helpful h, note that $\ell \mapsto \ell^\infty(A_n)$ is relatively weak* continuous and hence there is a relatively weak* open $L \subseteq \Delta(S)$ containing ℓ_n such that $\ell^\infty(A_n) > \varepsilon$ for all $\ell \in L$. Since $\ell_n \in \operatorname{supp}\mu$, $\mu(L) > 0$. Take any $a, b \in u(X)$ and define h by

$$u(h(\omega)) = a \text{ if } \omega \in \Psi^{-1}(L), \text{ and } u(h(\omega)) = b \text{ otherwise.}$$

Then, the left-hand side of (13) reduces to

$$\mu(L)\phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)a\right) + (1-\mu(L))\phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)b\right),$$

and the right-hand side of (13) becomes

$$\begin{aligned}&\int_L \phi\left(\alpha\ell^\infty(A_n) + (1-\alpha)a\right)d\mu(\ell) + \int_{\Delta(S)\setminus L} \phi\left(\alpha\ell^\infty(A_n) + (1-\alpha)b\right)d\mu(\ell)\\ \geq&\mu(L)\phi(\alpha\varepsilon + (1-\alpha)a) + (1-\mu(L))\phi((1-\alpha)b).\end{aligned}$$

Therefore, (13) implies

$$\begin{aligned}&(1-\mu(L))\left[\phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)b\right) - \phi((1-\alpha)b)\right]\\ \geq&\mu(L)\left[\phi(\alpha\varepsilon + (1-\alpha)a) - \phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)a\right)\right].\end{aligned}$$

Then,

$$\mu(L) \leq (1-\mu(L))\frac{\phi\left(\alpha\frac{\varepsilon}{2} + (1-\alpha)b\right) - \phi((1-\alpha)b)}{\phi(\alpha\varepsilon + (1-\alpha)a) - \phi\left(\alpha\frac{\varepsilon}{2} + (1-\alpha)a\right)}.$$

Since α, a and b were arbitrary, we have $\mu(L) \leq (1-\mu(L))K$ where

$$K = \inf\left\{\frac{\phi(a'+\delta) - \phi(a')}{\phi(b'+\delta) - \phi(b')} : a', b', a'+\delta, b'+\delta \in u(X), 0 < \delta \leq \tfrac{\varepsilon}{2}\right\}.$$

Recall that $\mu(L) > 0$ and hence $K > 0$. Thus, to show a contradiction, it suffices to show that $K = 0$. Let $\rho(t, t') = \left[\phi(t') - \phi(t)\right]/(t'-t)$. Assume the lower inequality in (i) fails – that is, for any $\gamma > 0$, $\rho(t, t') < \gamma$ for some $t < t' \in u(X)$. (The case where the upper inequality in (i) fails can be proved similarly.) Thus, for any $\delta \in (0, t'-t]$, there is $t'' \in u(X)$ such that $\rho(t'', t''+\delta) < \gamma$, because otherwise $\rho(t, t') < \gamma$ can't be true. Next take any $r < r' \in u(X)$ and let $\bar{\rho} = \rho(r, r') > 0$. By similar reasoning, for any $\delta \in (0, r'-r] > 0$, there is $r'' \in u(X)$ such that $\rho(r'', r''+\delta) \geq \bar{\rho}$. Thus,

$$\inf\left\{\frac{\rho(t'', t''+\delta)}{\rho(r'', r''+\delta)} : t'', r'', t''+\delta, r''+\delta \in u(X), 0 < \delta \leq \min[\tfrac{\varepsilon}{2}, r'-r, t'-t]\right\} = 0.$$

This infimum is at least K, thus $K = 0$, a contradiction. Because there are $m, M > 0$ such that $m|a-b| \leq |\phi(a) - \phi(b)| \leq M|a-b|$ for all $a, b \in u(X)$, we can apply Klibanoff et al. (2014, Theorem 4.3) to conclude that $R(\succsim) = \operatorname{supp}\mu$ as desired.

A.6. Proof of Theorem 4

We start by showing the necessity of Monotone Continuity of $\succsim^*$ and Wakker's axioms on $\mathcal{F}^\Psi$. Ghirardato and Siniscalchi (2010) show necessity of Cauchy continuity. Necessity of the remaining axioms is straightforward.

Monotone Continuity of $\succsim^*$: Case (i) can be treated in the same way as in the proof of Theorem 3.

Turn to the case where (ii) holds, so that $\operatorname{supp}\mu$ is finite. Again suppose $A_n \searrow \emptyset$ and $x \succ x' \succ x''$. Since $\operatorname{supp}\mu$ is finite, $\sup_{\ell\in\operatorname{supp}\mu} \ell^\infty(A_n) \to 0$. Thus, for $\varepsilon > 0$ satisfying $u\left(x'\right) > \varepsilon u(x) + (1-\varepsilon)\, u\left(x''\right)$, there is $n > 0$ such that $\ell^\infty(A_n) < \varepsilon$ for all $\ell \in \operatorname{supp}\mu$. This implies

$$\begin{aligned}
&U\left(\alpha x' + (1-\alpha)\, h\right) - U\left(\alpha x A_n x'' + (1-\alpha)\, h\right)\\
&= \int \phi\left(\alpha u\left(x'\right) + (1-\alpha)\int u(h)\, d\ell^\infty\right)\\
&- \phi\left(\alpha\left(\ell^\infty(A_n)\, u(x) + \left(1-\ell^\infty(A_n)\right) u\left(x''\right)\right) + (1-\alpha)\int u(h)\, d\ell^\infty\right) d\mu(\ell)\\
&\geq 0,
\end{aligned}$$

for all $\alpha \in [0,1]$, $h \in \mathcal{F}$, and $\ell \in \operatorname{supp}\mu$. Therefore, $x' \succsim^* x A_n x''$ and Monotone Continuity of $\succsim^*$ holds.

The fact that Wakker's axioms are satisfied on $\mathcal{F}^\Psi$ follows by the same reasoning as in the proof of Theorem 3. Note that Wakker's pointwise monotonicity axiom (see Wakker, 1989, Definition V.4.1) is implied by our Axiom 2.

Now turn to sufficiency. By the same reasoning as in the proof of Theorem 3, we can identify the set of acts $\mathcal{F}^\Psi$ with the set of "second-order" acts $F_{\Delta(S)} = \{f : \Delta(S) \to X : |f(S)| < \infty\}$. Indeed, for any $f \in \mathcal{F}^\Psi$ we can define the second order act $\ell \mapsto f \circ \Psi^{-1}(\ell)$. Conversely, using the same reasoning as in the proof of Theorem 3, for any $a \in F_{\Delta(S)}$ we can find $f \in \mathcal{F}^\Psi$ such that $f \circ \Psi^{-1}(\ell) = a$. It follows that we can define a preference relation on $F_{\Delta(S)}$ which satisfies the axioms of Wakker (1989). By Wakker (1989, Theorem V.6.1), $\succsim$ on $\mathcal{F}^\Psi$ can be represented by

$$V(f) = \int_{\Delta(S)} v\left(f \circ \Psi^{-1}(\ell)\right) d\mu(\ell),$$

for a wc continuous v on X and a countably additive measure $\mu \in \Delta(\Delta(S))$. Since $\succsim$ on X is vN-M, there is a mixture linear function u on X, representing $\succsim$ on X. Thus, $v = \phi \circ u$ for some strictly increasing function ϕ on $u(X)$. By Mixture Continuity of $\succsim$, $\alpha \mapsto u(\alpha x + (1-\alpha)\, y)$ is continuous on $[0,1]$. Since v is wc continuous, ϕ is continuous. Moreover, $u = \phi^{-1} \circ v$ is wc-continuous. Non-triviality implies u is non-constant.

Note that, for $f \in \mathcal{F}^\Psi$,

$$u\left(f \circ \Psi^{-1}(\ell)\right) = \int_{\Psi^{-1}(\ell)} u(f)\, d\ell^\infty = \int_{S^\infty} u(f)\, d\ell^\infty.$$

Thus,

$$V(f) = \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)\, d\ell^\infty\right) d\mu(\ell) \text{ for } f \in \mathcal{F}^\Psi.$$

It remains to extend to the entire $\mathcal{F}$. This can be done in the same way as in the proof of Theorem 3.

To complete sufficiency, we assume $\operatorname{supp}\mu$ is infinite and show (i) in the statement holds. The proof of Theorem 3 shows this is implied by Monotone Continuity of $\succsim^*$.

Uniqueness of u up to positive affine transformations is standard. Uniqueness of μ and ϕ follows by the construction – expected utility preference on acts in $\mathcal{F}^\Psi$ uniquely pin down μ and, when $\operatorname{supp}\mu$ is non-singleton, make ϕ unique up to positive affine transformations given a normalization of u. Finally, as either there are $m, M > 0$ such that $m\,|a-b| \leq |\phi(a) - \phi(b)| \leq M\,|a-b|$ for all $a, b \in u(X)$ or $\operatorname{supp}\mu$ is finite, we can again apply Klibanoff et al. (2014, Theorem 4.3) to conclude that $R(\succsim) = \operatorname{supp}\mu$.

A.7. A version of Theorem 3 with a partially atomic measure

Define the set of atoms in $\Delta(S)$ as

$$\mathcal{A} = \left\{\ell \in \Delta(S) : \Psi^{-1}(\ell) \text{ is non-null}\right\}.$$

Consider the following weakening of P6:

Axiom 15 *(Unlikely Atoms).* There exist $x, y \in X$ with $x \succ y$ such that $y\Psi^{-1}(\mathcal{A})x \succsim x\Psi^{-1}(\mathcal{A})y$.

In words, this axiom restricts the likelihood of the set of atoms of long-run frequencies.

Theorem 8. *$\succsim$ satisfies Axioms 1–7, P2-P4, Unlikely Atoms, Pointwise Continuity and Cauchy Continuity if and only if there is a non-constant vNM utility function $u : X \to \mathbb{R}$, a strictly increasing continuous function $\phi : u(X) \to \mathbb{R}$ such that there are $m, M > 0$ with $m|x-y| \leq |\phi(x) - \phi(y)| \leq M|x-y|$ for every $x, y \in u(X)$ and a Borel probability measure $\mu \in \Delta(\Delta(S))$ such that $\mu(\mathcal{A}) \leq \frac{1}{2}$ and*

$$U(f) = \int_{\Delta(S)} \phi\left(\int u(f)d\ell^\infty\right)\mu(\ell),$$

represents $\succsim$. Moreover, μ is unique, $R(\succsim) = \operatorname{supp}\mu$, u is unique up to a positive affine transformation, and, given a normalization of u, ϕ is unique up to positive affine transformations.

A.7.1. Proof of Theorem 8

The proof proceeds as in Section A.5. The only difference is that we invoke Stanca (2020, Theorem 6) instead of Stanca (2020, Theorem 5) when deriving μ. It is immediate that Unlikely Atoms is equivalent to the condition $\mu(\mathcal{A}) \leq \frac{1}{2}$.

A.8. Proof of Proposition 2

If either the support of μ is a singleton or ϕ is linear, (7) is (an increasing transformation of) a subjective expected utility functional and therefore satisfies Independence. For the other direction, argue by contradiction. Suppose that the preference satisfies Independence, ϕ is not linear and the support of μ contains at least two elements. Then there is a set $E \subset \Delta(S)$ such that $\mu(E) \in (0, 1)$. Independence plus Axioms 1–7 implies that preferences have an SEU representation with the same vNM utility function u as in the smooth ambiguity representation (7).

Denote the unique probability measure in this SEU representation by η. By Event Symmetry and the de Finetti theorem, η is symmetric and therefore there is a unique Borel probability measure $\lambda \in \Delta(\Delta(S))$ such that,

$$\eta(A) = \int_{\Delta(S)} \ell^{\infty}(A) d\lambda(\ell)$$

for all measurable $A \subseteq S^{\infty}$. Thus

$$\int_{\Delta(S)} \int_{S^{\infty}} u(f) d\ell^{\infty}(\omega) d\lambda(\ell)$$

represents the same preference as (7). In particular, they represent the same subjective expected utility preference when restricted to acts in $\mathcal{F}^{\Psi}$. Therefore

$$\int_{\Delta(S)} u(f(\Psi^{-1}(\ell)) d\lambda(\ell)$$

and

$$\int_{\Delta(S)} \phi \circ u(f(\Psi^{-1}(\ell)) d\mu(\ell)$$

represent the same SEU preferences over acts in $\mathcal{F}^{\Psi}$. Since $\mu(E) \in (0, 1)$ implies that μ has non-singleton support, the uniqueness properties of an SEU representation yield that $\phi \circ u$ must be a positive affine transformation of u. This contradicts the non-linearity of ϕ and completes the proof.

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