

Uniqueness of Recursive Utility with Unbounded Consumption and Non-Expected Utility*

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Abstract

We derive sufficient conditions for the existence, non-existence, uniqueness and global attractivity of stochastic recursive utilities. Our approach uses abstract order-theoretic fixed-point arguments, allowing us to accommodate a large class of Markov consumption-growth models with unbounded per-period consumption and a broad range of recursive preferences, including non-expected utility specifications. We illustrate the usefulness of the framework through macro-finance applications to widely used consumption-growth models under several preference specifications, including Epstein–Zin preferences, multiple priors, smooth ambiguity, and Hansen–Sargent robust-control preferences. In each application, our general conditions reduce to a simple, economically interpretable parameter test that shows how fundamental parameters influence existence and uniqueness. While uniqueness of recursive utility can be difficult to guarantee in unbounded environments, in a Markov setting with non-unit elasticity we establish uniqueness within the class of recursive utilities having a so-called separability property.

Keywords: Stochastic recursive utility, unbounded consumption, non-expected utility, fixed points, macro-finance.

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1 Introduction

Recursive utility plays an important role in macroeconomics and finance because it provides a tractable way to separate intertemporal substitution from risk aversion

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and, more generally, from attitudes toward uncertainty. This flexibility has led to applications across a wide range of topics, from household finance [13] to models that combine economic and climate risks [12]. Following the seminal contributions of Weil [41, 42], recursive utility has become a standard tool in macro-finance for incorporating ambiguity, multiple priors, robust control, and other departures from expected utility; see Backus, Routledge, and Zin [3] for a review.

This flexibility comes at a cost. Recursive utility is a fixed point of a nonlinear stochastic operator, so neither existence nor uniqueness can be taken for granted. Moreover, applications often feature non-expected utility or ambiguity, which further complicates the analysis. Hence, before using a recursive preference specification in applications, one must know whether the recursion has a solution and whether that solution is unique. These questions are not merely technical. If multiple solutions exist, the model may generate ambiguous welfare rankings or indeterminate asset-pricing implications.

Following earlier contributions, including Marinacci and Montrucchio [30], one strand of the literature has relied on finding metrics to recover contraction properties. These methods, however, typically require restrictions that are strong for macroeconomic applications, such as consumption being bounded at every date. Such restrictions are automatically violated by standard consumption processes with normally distributed error terms.

This paper aims to fill this gap by providing general conditions for the existence, non-existence, uniqueness, and global attractivity of stochastic recursive utilities with unbounded consumption. Our approach relies on fixed-point methods in ordered spaces, especially the techniques of Kantorovich [24] and Krasnosel'skii [27], as already used by a few authors (see, for example, Christensen [14]).

The key novelty is our uniqueness argument. A result by Marinacci and Montrucchio [31, Theorem 1] allows us to establish uniqueness using a method that is comparatively simple relative to existing approaches. It is based on inspecting the behavior of the concave operator on the so-called *lower perimeter* of its domain. This approach is often simpler than computing the spectral radius of a subgradient/supergradient of the convex/concave operator, which is the alternative route to uniqueness (see for instance [2] and [14]).

Recursive utility consists of two main ingredients: a *time aggregator* and a *certainty equivalent*. The time aggregator determines how current consumption is combined with future continuation utility, and therefore governs the elasticity of intertemporal substitution (EIS). The certainty equivalent summarizes the agent's evaluation of uncertainty about future continuation values, capturing risk attitudes and, more generally, attitudes toward uncertainty. We focus on the two aggregators most commonly used in applications, CES and Cobb–Douglas. Our assumptions on certainty equivalents are deliberately minimal: monotonicity and subhomogeneity. These conditions have natural behavioral interpretations. For instance, subhomogeneity is related to increasing relative risk aversion (see [30]).

Our sufficient conditions for existence have a simple economic interpretation. For CES aggregators, they depend on an *adjusted consumption growth measure* that incorporates attitudes toward uncertainty, discounting, and intertemporal substitutability. When the EIS is below one, existence requires this adjusted measure to be uniformly bounded away from zero; that is, consumption cannot grow too slowly.

In contrast, when the EIS is above one, existence requires the adjusted measure to be uniformly bounded from above; that is, consumption cannot grow too fast. In the unit-EIS case, a special case of the Cobb–Douglas aggregator, the analogous condition is the integrability of consumption adjusted for attitudes toward uncertainty. The non-existence results mirror these conditions: recursive utility may fail to exist if adjusted growth is uniformly too low or uniformly too high.

These existence results may yield more than one fixed point. The key assumption that delivers uniqueness of recursive utility is order concavity of the certainty equivalent, which has a natural economic interpretation in terms of risk aversion or ambiguity aversion. Finally, we show that uniqueness also implies global attractivity: starting from any process comparable to consumption, repeated application of the recursive operator converges, in the relevant sense, to the unique fixed point.

Thus far, our analysis has been deliberately abstract. We now specialize the results to a standard Markov environment that covers many consumption growth processes used in applications. The economy has a Markov state variable, which represents an economically relevant variable such as the growth regime or disaster risk. Consumption growth is allowed to depend on the current state, the next state, and an independent shock.

In this Markov setting, when the certainty equivalent is also positively homogeneous and stationary, the problem can be simplified by considering recursive utilities that take a particularly simple form: current consumption multiplied by a function of the current state. We call such recursive utilities *separable*. This class of recursive utilities is useful because it replaces the original problem of solving for an entire utility process with the simpler problem of solving for a single function of the state. Notably, with non-unit EIS, our conditions imply that whenever a separable recursive utility exists, it is unique. By contrast, at unit EIS multiple separable recursive utilities may coexist, and for Epstein–Zin preferences we characterize the set of separable recursive utilities.

We illustrate the usefulness of the framework through applications to widely used consumption-growth models in macro-finance under several preference specifications. We begin with the classic Mehra–Prescott regime switching economy. In the parameter range considered by Weil [41], we show that our conditions for existence and uniqueness of recursive utility are always satisfied. For certain parameters in this range, however, relatively small changes in consumption growth rates are enough to obtain non-existence: an increase of about 5% in a high-EIS case and a reduction of about 3% in a low-EIS case.

The second application is a two-state hidden-regime model, in which consumption

growth depends on an unobserved regime. Models of this type are standard in macro-finance; see, for instance, Lettau, Ludvigson, and Wachter [28] and Ju and Miao [23]. Under the Ju–Miao calibration, our conditions imply existence and uniqueness under smooth ambiguity (non-unit EIS), Hansen–Sargent robust control (unit EIS), and multiple priors (infinite ambiguity aversion). Notably, for the smooth-ambiguity specification, uniqueness of recursive utility had not been established. Our results close this gap by proving uniqueness under the Ju–Miao calibration.

To illustrate the non-existence conditions, we consider a counterfactual with a less persistent low-growth regime. Under smooth ambiguity, an increase of about 8.3% in gross growth in both regimes is sufficient for non-existence. Under multiple priors, by contrast, a much larger increase of about 15.5% in both growth rates is required to satisfy the sufficient condition for non-existence.

The third application is a rare-disaster model in which consumption growth is exposed to persistent disaster risk, as in Wachter [40] and Bidder and Smith [7]. In this environment, the conditions guaranteeing uniqueness of recursive utility may fail even though our existence conditions are satisfied. In particular, with unit EIS the rare-disaster model can admit multiple separable recursive utilities. Surprisingly, with non-unit EIS our results imply uniqueness within the class of separable recursive utilities, even though they do not rule out other non-separable recursive utilities.

Finally, we apply our characterization of separable recursive utilities at unit EIS to the constant-volatility consumption specification of Bansal and Yaron [5]. This yields two distinct utilities belonging to disjoint cones (Section 4.7).

The paper is organized as follows. Section 2 introduces the mathematical framework. Section 3 establishes the main existence, uniqueness, global-attractivity, and non-existence results. Section 4 applies the general framework to standard macro-finance models. Section 5 concludes. Appendix A contains the proofs and auxiliary theoretical results, while the Supplemental Appendix (Appendix S) provides additional derivations and calculations for the applications.

Related literature. The literature on recursive preferences with unbounded consumption remains limited, particularly with respect to uniqueness, which is often established only in special cases. The contributions most closely related to ours are Borovička and Stachurski [10], Hansen and Scheinkman [20, 21], Christensen [14], and Pohl, Schmedders, and Wilms [36]. We return in Section 3.6 to the more technical connection between their approaches and the Epstein–Zin special case of our framework.

Borovička and Stachurski [10] obtain necessary and sufficient conditions for existence and uniqueness in the Epstein–Zin model under the assumption that the state space is compact. In contrast, our conditions apply to a broader class of certainty equivalents and time aggregators, including non-expected-utility specifications. Moreover, in certain cases, we can allow for an unbounded state space.

Hansen and Scheinkman [20, 21] develop a spectral approach to Epstein–Zin re-

cursive utility, establishing existence and, under additional restrictions, uniqueness. Our approach accommodates a broader class of certainty equivalents.

Christensen [14] studies recursive utility without boundedness assumptions using exponential-Orlicz spaces, which are related to thin-tailed conditions. His framework allows noncompact state spaces and unbounded consumption, and provides existence and uniqueness results for several important recursive specifications with unit EIS. By contrast, our approach yields existence, uniqueness, and global attractivity for a broader class of preferences.

Pohl, Schmedders, and Wilms [36] derive existence and non-existence conditions for several recursive preference specifications, including Epstein–Zin and smooth ambiguity. Our existence conditions do not rely on Jensen’s inequality. For Epstein–Zin preferences, they neither imply nor are implied by the sufficient conditions obtained from their explicit Jensen and Minkowski bounds.

A related dynamic-programming literature studies Bellman equations with recursive utility. Le Van and Vailakis [29] analyze recursive utility in optimal growth models with unbounded returns, and Martins-da-Rocha and Vailakis [32] use local-contraction methods. Bloise and Vailakis [9], Balbus [4], Ren and Stachurski [37], and Bloise, Le Van, and Vailakis [8] use monotone, concave, or Tarski-type methods in related recursive-utility and dynamic-programming problems. In particular, Ren and Stachurski [37] use fixed-point arguments due to Du [15] (see in particular his Theorem 3.1). This fixed-point argument requires the positive cone of the relevant space to have nonempty interior. In our L^1 setting, the positive cone has empty interior.

Finally, observe that solving forward-looking equations with unbounded variables of the type studied here is also relevant for general equilibrium models, as they are required to characterize equilibrium allocations and prices (e.g., see Beker and Chen [6]).

2 Mathematical preliminaries

Stochastic processes. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space equipped with an increasing filtration $\Sigma = (\Sigma_t)_{t \in T}$, where $T = \{0, 1, 2, \dots\}$. A process $X = (X_t)_{t \in T}$ is adapted if X_t is Σ_t -measurable for all $t \in T$. Denote by L the set of all finitely valued adapted processes, $L_+ \subset L$ the nonnegative processes and L_{++} the strictly positive ones.

It is conceptually useful to regard a discrete-time adapted process as a measurable function defined on a suitable σ -algebra (see also Appendix A of [30]). It is enough to introduce the σ -algebra $\mathcal{F}$ in $\Omega \times T$, defined as $A \in \mathcal{F}$ if and only if the t -section $A_t \in \Sigma_t$ for all $t \geq 0$.

In what follows, some subspaces of L_+ will be relevant. Given a scalar $0 < \delta < 1$, consider the discrete measure $c_\delta(\{t\}) = \delta^t$ for $t \in T$ and the corresponding L^1 -space $L^1(\Omega \times T, \mathcal{F}, \mathbb{P} \otimes c_\delta) \subset L$, denoted by $L^1(\delta)$ for simplicity. The norm in $L^1(\delta)$ is

given by

$$\|X\| = \int |X(\omega, t)| d(\mathbb{P} \otimes c_\delta)$$

where X is an equivalence class of measurable functions rather than a single function. We use the same symbol to denote the L^1 -norms of the functions X_t , if no confusion arises. Namely,

$$\|X_t\| = \int |X_t| d\mathbb{P} = \mathbb{E}[|X_t|] \quad \text{for each } t \in T.$$

By Fubini's theorem we have the relation¹

$$\|X\| = \sum_{t \in T} \|X_t\| \delta^t.$$

The order relation $X \leq Y$ in L , as well as in $L^1(\delta)$, means $X_t \leq Y_t$, a.s., for all $t \in T$. The cone $L_+^1(\delta)$ consists of the elements $X \in L^1(\delta)$ satisfying $X \geq 0$. The cone $L_{++}^1(\delta)$ consists of the elements $X > 0$, namely $X_t > 0$, a.s., for all t . We will also use the stronger order relation $Y \gg X$, meaning that $Y \geq X + \varepsilon$ for some $\varepsilon > 0$.

The space $L^1(\delta)$ equipped with the order $\leq$ is a Banach lattice (i.e., a Riesz space endowed with a complete lattice norm, see for instance [1, 35]). It has the property of being Dedekind σ -complete². This entails the so-called *monotone convergence property*. In symbols, if $X_n \uparrow \leq X$ for some $X \in L^1(\delta)$, then there is an element $X^* \in L^1(\delta)$ such that $X_n \uparrow X^* = \sup_n X_n$ and the convergence holds in the space $L^1(\delta)$. This last property is just a consequence of the monotone convergence theorem. In the Russian literature ordered vector spaces having these properties are said to have a *regular positive cone* (see [27, Section 1.5]).

Given an element $X \in L_+$, $\mathcal{K}(X)$ denotes the convex cone

$$\mathcal{K}(X) = \bigcup_{0 < \lambda < \mu < \infty} [\lambda X, \mu X].$$

An element Y belongs to $\mathcal{K}(X)$ if and only if $\lambda X \leq Y \leq \mu X$ holds for some $\lambda, \mu > 0$. For future reference, notice that in this case there exist a least $\bar{\mu}$ and a greatest $\bar{\lambda}$ for which one has $\bar{\lambda}X \leq Y \leq \bar{\mu}X$.³

To gain economic intuition for the condition $Y \in \mathcal{K}(X)$, observe that it implies that Y and X have comparable long-run growth rates: more precisely, these rates do not diverge from one another on average. The cone $\mathcal{K}(C)$ of consumption processes comparable with C will play an important role throughout the paper. Indeed, it will be the natural domain for the recursive preferences associated with the consumption process C . Let us add some simple properties of it. If $C \in L_{++}$, then $\mathcal{K}(C) \subset L_{++}$. If $C \in L_{++}^1(\delta)$, then $\mathcal{K}(C) \subset L_{++}^1(\delta)$.

¹This $L^1(\delta)$ -norm is the L^1 -counterpart of weighted supnorms adopted by Boyd [11] and [30].

²Indeed, it is Dedekind complete too (see for instance [34, Proposition II.4.1]), but the completeness for countable sets suffices for our purposes, as will become apparent later on.

³This property is true in any Archimedean ordered vector space.

Aggregators. Consider the CES aggregator $W : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_+$ given by

$$W(x, y) = \left((1 - \beta)x^r + \beta y^r \right)^{1/r}, \quad x, y > 0$$

where $0 < \beta < 1$ and $r \in (-\infty, 0) \cup (0, 1)$. The aggregator W is monotone, positively homogeneous and concave. Clearly, for $r \in (0, 1)$ it can be extended continuously on $x, y \geq 0$. Note that the economic counterpart of r is the *elasticity of intertemporal substitution*, defined by

$$\text{EIS} := \frac{1}{1 - r}.$$

Hence, $r < 0 \iff 0 < \text{EIS} < 1$, while $0 < r < 1 \iff \text{EIS} > 1$. Moreover, observe that the limit case $r \rightarrow 0$ yields the Cobb–Douglas aggregator $W(x, y) = x^{1-\beta}y^\beta$, corresponding to unit elasticity of intertemporal substitution, $\text{EIS} = 1$.

Certainty equivalents. The certainty equivalent operator $\mathcal{M}$, in general, maps a portion of L_+ into L_+ . The notation $\mathcal{M}_t(X)$ denotes the t -th component of $\mathcal{M}(X)$. At present, the certainty equivalent will not be entirely specified. We list below some classes of certainty operators that will be used below.

- i) $\mathcal{M}$ is *monotone*, i.e., $X \leq Y \implies \mathcal{M}X \leq \mathcal{M}Y$
- ii) $\mathcal{M}$ is *σ -order continuous*, i.e., $X^n \uparrow X \implies \mathcal{M}(X^n) \uparrow \mathcal{M}(X)$, as well as $X^n \downarrow X \implies \mathcal{M}(X^n) \downarrow \mathcal{M}(X)$
- iii) $\mathcal{M}$ is *subhomogeneous*, i.e., $\mathcal{M}(\alpha X) \geq \alpha \mathcal{M}(X)$ for $\alpha \in (0, 1) \iff \mathcal{M}(\lambda X) \leq \lambda \mathcal{M}(X)$ for $\lambda > 1$.
- iv) $\mathcal{M}$ is *order concave*, i.e.,

$$\mathcal{M}(\alpha X_1 + (1 - \alpha) X_2) \geq \alpha \mathcal{M}(X_1) + (1 - \alpha) \mathcal{M}(X_2)$$

holds for every $\alpha \in [0, 1]$ and $0 < X_1 \leq X_2$. Order convexity is defined analogously. Note that (ii) implies (i).

Remark 1. A broad class of subhomogeneous operators is given by the so-called *quasi-arithmetic means*, i.e., $\mathcal{M}_t(X) = \phi^{-1}(\mathbb{E}_t(\phi \circ X_{t+1}))$ where ϕ is a twice-differentiable and strictly monotone function that exhibits increasing relative risk aversion (IRRA); see [30, Proposition 7]. In the following, a central role will be played by the Kreps–Porteus case $\phi(t) = t^s$, i.e., $\mathcal{M}_t(X) = \left(\mathbb{E}_t(X_{t+1}^s) \right)^{1/s}$ where $1 - s$ is the coefficient of relative risk aversion,

$$\text{RRA} := 1 - s.$$

We refer also to the discussion in Section 3.6 and Proposition 17. Its Markovian counterpart is the Epstein–Zin certainty equivalent treated in subsection 4.1 (see (23)).

Recursive utility. Recursive utilities will be identified with the fixed points of an operator acting on the space $L^1(\delta)$. Given a consumption process $C \in L_{++}$, the operator $\mathbf{T}$ under study is⁴

$$\mathbf{T}X := W(C, \mathcal{M}(X)). \quad (1)$$

A recursive utility is a fixed point U of $\mathbf{T}$, i.e.,

$$U = \mathbf{T}(U) \iff U_t = W(C_t, \mathcal{M}_t(U))$$

for all $t \in T$.

In the following we will restrict our analysis to the most relevant certainty operators for which $\mathcal{M}_t(U) \equiv \mathcal{M}_t(U_{t+1})$. Written explicitly, the recursive utilities U_t solve the forward-looking difference equation

$$U_t = [(1 - \beta)C_t^r + \beta(\mathcal{M}_t(U_{t+1}))^r]^{1/r} \text{ for } t \in T \quad (2)$$

Here C_t is the current consumption and the stream $(U_t)_{t \geq 0}$ associated with the consumption process C is known in the macroeconomics literature as the *continuation value* (see for instance Hansen et al. [18]). In this setting, β is the *subjective discount factor*, while $r = 1 - \text{EIS}^{-1}$ is the parameter associated with the elasticity of intertemporal substitution.

We note that the operator $\mathbf{T}$ is subhomogeneous, provided $\mathcal{M}$ is. Actually, since our aggregator function W is monotone and positively homogeneous, we have, for $\alpha \in (0, 1)$

$$\mathbf{T}(\alpha X) = W(C, \mathcal{M}(\alpha X)) \geq W(\alpha C, \alpha \mathcal{M}(X)) = \alpha W(C, \mathcal{M}(X)) = \alpha \mathbf{T}(X)$$

Moreover, as W is continuous, if $\mathcal{M}$ is σ -order continuous then $\mathbf{T}$ enjoys the same property.

Let $\mathcal{M}$ be monotone and subhomogeneous. If $\mathcal{M}(C) \in L_+$, then $\mathcal{K}(C) \subset L_+$. Indeed, if $X \in \mathcal{K}(C)$, then $X \leq \lambda C$ for some $\lambda > 1$. Hence, under our hypotheses

$$\mathcal{M}(X) \leq \mathcal{M}(\lambda C) \leq \lambda \mathcal{M}(C) \quad (3)$$

and so $\mathcal{M}(X) \in L_+$.

3 Existence and uniqueness

In this section we study in detail the recursive preference (2), namely the fixed points of the operator $\mathbf{T}$ of equation (1). Since the sign of r determines the relevant existence and uniqueness conditions, it is convenient to distinguish the low-EIS case $r < 0$ ($\text{EIS} < 1$) from the high-EIS case $0 < r < 1$ ($\text{EIS} > 1$). We then consider the Cobb–Douglas limit $r = 0$ ($\text{EIS} = 1$), before establishing global attractivity and the related non-existence results.

⁴Indeed $\mathbf{T}$ depends on the process C , and so one should write more correctly $\mathbf{T}_C$. Analogously a fixed point U of $\mathbf{T}_C$ should be labelled as U_C .

3.1 Low EIS: $r < 0$

Since the function $\varphi(t) = [(1 - \beta)x^r + t]^{1/r}$ is decreasing,

$$(1 - \beta)^{1/r}x = \varphi(0) \geq [(1 - \beta)x^r + \beta y^r]^{1/r} = W(x, y)$$

and so we have the upper bound

$$W(x, y) \leq (1 - \beta)^{1/r}x \quad (4)$$

To get results on the fixed points of $\mathbf{T}$ we impose restrictions on the consumption processes under analysis.

Lemma 2. *Let $\mathcal{M}$ be monotone and subhomogeneous and $C \in L_{++}$. Suppose that the following two assumptions hold:*

- a) $\mathcal{M}(C) \in L_+$;
- b)

$$\frac{\mathcal{M}(C)}{C} \gg \beta^{\frac{1}{|r|}}. \quad (5)$$

Then

$$\frac{\mathbf{T}(\varepsilon C)}{\varepsilon C} \gg 1 \quad (6)$$

holds for all $\varepsilon > 0$ sufficiently small.

Observe that (6) implies that $\mathbf{T}(\varepsilon C) \geq \varepsilon C$. Moreover (4) implies that $\mathbf{T}(X) \leq (1 - \beta)^{1/r}C$ whenever $\mathbf{T}(X)$ is well defined. Therefore, under the hypothesis of this Lemma, we deduce that $\mathbf{T}$ maps the interval $[\varepsilon C, (1 - \beta)^{1/r}C]$ into itself for all sufficiently small $\varepsilon > 0$.

This is the main ingredient to get the next result on the existence of fixed points. Its proof uses an argument based on Kantorovich's fixed-point theorem [24, p. 68]; see also [27, Th. 4.1(b)] and [14, Prop. 3.1 (i)].

Proposition 3. *Let $r < 0$. Assume that (i)-(ii)-(iii) hold for the certainty operator $\mathcal{M}$. Let $C \in L_{++}^1(\delta)$ for some $\delta > 0$ and $\mathcal{M}(C) \in L_+$. If*

$$\frac{\mathcal{M}(C)}{C} \gg \beta^{\frac{1}{|r|}}, \quad (7)$$

then $\mathbf{T} : \mathcal{K}(C) \rightarrow \mathcal{K}(C)$. More precisely, for every $X \in \mathcal{K}(C)$ there exists an $\varepsilon > 0$ such that

$$\varepsilon C \leq \mathbf{T}(X) \leq (1 - \beta)^{1/r}C \quad (8)$$

Moreover, the set of the fixed points of $\mathbf{T}$ in $\mathcal{K}(C)$ is not empty and contains the greatest fixed point.

A few comments are in order. The condition $C \in L_{++}^1(\delta)$ is a very mild restriction on the growth of the paths of consumption as δ approaches 0. Though the value δ does not appear in condition (7), its positivity is crucial to guarantee the Dedekind σ -completeness of the space.

As will be seen in the proof, a consequence of the fact that $\mathbf{T}$ maps $[\varepsilon C, (1 - \beta)^{1/r} C]$ into itself is that $\mathbf{T}$ also has a least fixed point U_*^ε in this interval; but it depends on the scalar ε and so the existence of a least fixed point in $\mathcal{K}(C)$ is not assured.

Condition (5) has a simple economic interpretation. It is equivalent to

$$\frac{1}{\beta} \left(\frac{\mathcal{M}(C)}{C} \right)^{|r|} \gg 1. \quad (9)$$

The ratio $\mathcal{M}(C)/C$ is the certainty-equivalent value of future consumption relative to current consumption. Hence the left-hand side is an adjusted measure of consumption growth that incorporates attitudes toward uncertainty, discounting, and intertemporal substitutability. When the EIS is below one, existence requires this adjusted growth measure to be uniformly large enough: consumption cannot grow too slowly.

3.1.1 Uniqueness

We need the concept of lower perimeter $\partial_\diamond A$, for a subset A of an ordered vector space, introduced by Marinacci and Montrucchio [31]. We refer to that paper for the general definition and focus here only on the lower perimeter $\partial_\diamond [\varepsilon C, (1 - \beta)^{1/r} C]$ of the interval $[\varepsilon C, (1 - \beta)^{1/r} C]$ of Proposition 3.

The elements of such a lower perimeter are⁵

$$\partial_\diamond [\varepsilon C, (1 - \beta)^{1/r} C] = \left\{ H \in [\varepsilon C, (1 - \beta)^{1/r} C] : \inf \frac{H}{C} = \varepsilon \right\} \quad (10)$$

Actually, by Proposition 4 of [31], the process $H \in \partial_\diamond [\varepsilon C, (1 - \beta)^{1/r} C]$ if and only if

$$\begin{aligned} \inf \frac{H - \varepsilon C}{((1 - \beta)^{1/r} - \varepsilon)C} &= 0 \iff \inf \frac{H}{((1 - \beta)^{1/r} - \varepsilon)C} = \frac{\varepsilon}{((1 - \beta)^{1/r} - \varepsilon)} \\ &\iff \inf \frac{H}{C} = \varepsilon \end{aligned}$$

We are in a position to establish sufficient conditions for uniqueness of the fixed points of $\mathbf{T}$. Observe that order concavity of $\mathcal{M}$ implies that $\mathbf{T}$ is order concave as well, as it is easy to check.

⁵Recall that all our random variables are indeed equivalence classes. Thus, $\inf_\omega Y(\omega)$ means $\inf \{x : \mathbb{P}(Y < x) > 0\}$.

Proposition 4. *Under the assumptions of Proposition 3, suppose in addition that $\mathcal{M}$ is order concave. Then the operator $\mathbf{T}$ has a unique fixed point in the cone $\mathcal{K}(C)$.*

This result shows that, once existence is guaranteed, no further growth restriction is needed for uniqueness in the case $r < 0$. Order concavity of the certainty equivalent is sufficient to obtain uniqueness. Economically, this means that sufficiently strong risk or uncertainty aversion rules out multiple continuation-value processes compatible with the same consumption stream. We now turn to the high-EIS case.

3.2 High EIS: $r \in (0, 1)$

In this case the CES aggregator W has the obvious lower bound

$$W(x, y) \geq (1 - \beta)^{1/r} x \quad (11)$$

that implies $\mathbf{T}(X) \geq (1 - \beta)^{1/r} C$.

An argument analogous to that used for the case $r < 0$ shows that the condition

$$\frac{\mathcal{M}(C)}{C} \ll \beta^{-1/r}$$

implies that

$$\mathbf{T}(\lambda C) \leq \lambda C \text{ for } \lambda > 1 \text{ large enough.}$$

Therefore, a proof similar to that of Proposition 3 delivers the next result.

Proposition 5. *Let $0 < r < 1$. Assume that (i)-(ii)-(iii) hold for the certainty operator $\mathcal{M}$. Let $C \in L_{++}^1(\delta)$ for some $\delta > 0$ and $\mathcal{M}(C) \in L_+$. If*

$$\frac{\mathcal{M}(C)}{C} \ll \beta^{-1/r}$$

then $\mathbf{T} : \mathcal{K}(C) \rightarrow \mathcal{K}(C)$. More precisely, for all $X \in \mathcal{K}(C)$ there exists a scalar $\lambda > 1$ for which

$$(1 - \beta)^{1/r} C \leq \mathbf{T}(X) \leq \lambda C.$$

Moreover, the set of the fixed points of $\mathbf{T}$ is not empty in $\mathcal{K}(C)$ and the least fixed point is included in it.

The analogue of (9) is here

$$\beta \left(\frac{\mathcal{M}(C)}{C} \right)^r \ll 1.$$

This condition also has a simple economic interpretation. The left-hand side is an adjusted measure of consumption growth that incorporates attitudes toward uncertainty, discounting, and intertemporal substitutability. When the EIS is above one,

existence requires this adjusted growth measure to be uniformly bounded from above: consumption cannot grow too fast.

In this case the lower perimeter is given by

$$\partial_{\diamond} \left[(1 - \beta)^{1/r} C, \lambda C \right] = \left\{ H \in \left[(1 - \beta)^{1/r} C, \lambda C \right] : \inf \frac{H}{C} = (1 - \beta)^{1/r} \right\}$$

Therefore for uniqueness it is required that $\mathbf{T} \left((1 - \beta)^{1/r} C \right) / C \gg (1 - \beta)^{1/r}$ together with the assumption of order concavity. Notice that $\mathbf{T} \left((1 - \beta)^{1/r} C \right) \gg (1 - \beta)^{1/r} C$ is equivalent to the condition $\mathcal{M}(C) / C \gg 0$. The next proposition formalizes these facts.

Proposition 6. *Let $0 < r < 1$. Assume that (i)-(ii)-(iii) hold for the certainty operator $\mathcal{M}$. Under the additional condition that $\mathcal{M}$ is order concave, $\mathbf{T}$ has a unique fixed point in $\mathcal{K}(C)$ if*

$$0 \ll \frac{\mathcal{M}(C)}{C} \ll \beta^{-1/r}$$

The key point of the proof of this result is that the relation $\mathbf{T} \left((1 - \beta)^{1/r} C \right) / C \gg (1 - \beta)^{1/r}$ is equivalent to $\mathcal{M}(C) / C \gg 0$, which in turn is nothing but the lower perimeter condition for uniqueness discussed earlier.

3.3 Unit EIS: $r = 0$ and Cobb–Douglas case

We now look at the Cobb–Douglas aggregator

$$W(x, y) = x^{\alpha} y^{\beta}$$

with $\alpha > 0$ and $\beta \in (0, 1)$. The specification $\alpha + \beta = 1$ is of some interest since it can be seen as the limit of the CES aggregator as $r \rightarrow 0$ (i.e., $\text{EIS} \rightarrow 1$). The associated operator is given by

$$\mathbf{T}(X) = C^{\alpha} [\mathcal{M}(X)]^{\beta} \tag{12}$$

Set $p = (1 - \alpha) / \beta$. Clearly $p \geq 1$ when $\alpha + \beta \leq 1$ and $p = 1$ if $\alpha + \beta = 1$.

The search for fixed points of (12) is quite simplified with respect to the CES case, thanks to a property which we are going to highlight in the following definition.

Definition 7. *An operator $\mathbf{T} : L_+ \rightarrow L_+$ is said to be q -subhomogeneous, $q \in (0, 1)$, if*

$$\mathbf{T}(\varepsilon X) \geq \varepsilon^q \mathbf{T}(X) \iff \mathbf{T}(\lambda X) \leq \lambda^q \mathbf{T}(X)$$

holds for all $X \in L_+$, $\varepsilon \in (0, 1)$ and $\lambda > 1$.

We refer to [31] for more details on this strengthened version of subhomogeneity. This immediately yields the following result. Its trivial proof is omitted.

Proposition 8. *If the certainty equivalent $\mathcal{M}$ is subhomogeneous, then the operator $\mathbf{T}$ in (12) is β -subhomogeneous with $\beta \in (0, 1)$.*

Unlike the CES case, the next proposition proves that existence and uniqueness in $\mathcal{K}(C)$ coincide, and concavity is not needed, thanks to Proposition 8.

Proposition 9. *Assume that (i)-(ii)-(iii) hold for $\mathcal{M}$. Then $\mathbf{T} : \mathcal{K}(C) \rightarrow \mathcal{K}(C)$ and the set of its fixed points is not empty in $\mathcal{K}(C)$ if and only if*

$$\mathcal{M}(C) \in \mathcal{K}(C^p). \quad (13)$$

In such a case $\mathbf{T}$ has exactly one fixed point in $\mathcal{K}(C)$.

If $r = 0$ then $p = 1$, and condition (13) reduces to $\mathcal{M}(C) \in \mathcal{K}(C)$. Thus, in the unit-EIS case, this growth condition guarantees the existence and uniqueness in $\mathcal{K}(C)$.

Remark 10. Uniqueness cannot hold in all of $L_{++}^1(\delta)$. Suppose that $\mathcal{M}$ is positively homogeneous. If U is a fixed point in $L_{++}^1(\delta)$ then $V(\lambda) = (\lambda^{\beta-t} U_t)$ is a fixed point of $\mathbf{T}$ in $L_{++}^1(\delta)$, for every $\lambda \in (0, 1]$. In other words, every fixed point generates a continuum of non-comparable fixed points. Indeed, $\mathcal{K}(V(\lambda)) \cap \mathcal{K}(V(\lambda_1)) = \emptyset$ for $\lambda \neq \lambda_1$. To see this, let $V \equiv V(\lambda)$ for short. Then

$$\begin{aligned} (\mathbf{T}V)_t &= C_t^\alpha [\mathcal{M}_t(V_{t+1})]^\beta = C_t^\alpha [\mathcal{M}_t(\lambda^{\beta-(t+1)} U_{t+1})]^\beta \\ &= \lambda^{\beta-t} C_t^\alpha [\mathcal{M}_t(U_{t+1})]^\beta = \lambda^{\beta-t} U_t = V_t \end{aligned}$$

Since $V(\lambda) \leq U$, $V(\lambda) \in L_{++}^1(\delta)$ and so it is a fixed point.

The condition $\mathcal{M}(C) \in \mathcal{K}(C)$ for $\alpha + \beta = 1$ is quite strong and, as we shall see later in Section 4.3, it may fail in economically relevant cases. This does not imply the non-existence of fixed points outside the cone $\mathcal{K}(C)$. Looking at the proof of Proposition 8, it is easy to realize that it can be generalized by replacing the cone $\mathcal{K}(C)$ with any other cone $\mathcal{K}(Z)$, with $Z \in L_{++}^1(\delta)$.

Proposition 11. *Assume that $\mathcal{M}$ satisfies (i)-(iii). Let $\alpha + \beta = 1$ and $Z \in L_{++}^1(\delta)$. Then $\mathbf{T} : \mathcal{K}(Z) \rightarrow \mathcal{K}(Z)$ and has a unique fixed point in $\mathcal{K}(Z)$ if and only if*

$$\mathcal{M}(Z) \in \mathcal{K}(C^{-\alpha/\beta} Z^{1/\beta}). \quad (14)$$

In other words, to each admissible cone $\mathcal{K}(Z)$ there corresponds a unique fixed point $U \in \mathcal{K}(Z)$, provided (14) holds.

For the Epstein–Zin certainty equivalent $\mathcal{M}_t(X) = (\mathbb{E}_t X_{t+1}^s)^{1/s}$, we can go further and characterize explicitly the processes Z satisfying (14). Indeed, condition (14) is

equivalent to the existence of an adapted process $B \in \mathcal{K}(\mathbf{1})$ such that (see the proof of Proposition 11)

$$\mathbb{E}_t [Z_{t+1}^s] = C_t^{-\alpha s/\beta} Z_t^{s/\beta} B_t. \quad (15)$$

It is worth remarking that, whenever in (15) the process Z is generated by a constant $B_t \equiv \bar{B} > 0$, we can calculate the unique recursive utility in $\mathcal{K}(Z)$ given by

$$U = \bar{B}^{\beta/(s\alpha)} Z. \quad (16)$$

3.4 Attractivity

Uniqueness of the fixed point implies that the iterates converge to it from any initial point. Recall that in a Riesz space a sequence (X_n) order converges to the point X , in symbols $X_n \dot{\rightarrow} X$, whenever

$$|X - X_n| \leq Z_n$$

holds for a nonnegative and decreasing sequence $Z_n \downarrow 0$. Clearly in L^1 spaces order convergence implies pointwise convergence a.s., as well as L_1 -convergence thanks to Beppo Levi's theorem.

Proposition 12. *Assume that $C \in L^1_{++}(\delta)$. Let $\mathbf{T} : \mathcal{K}(C) \rightarrow \mathcal{K}(C)$ be the operator of Proposition 4 or Proposition 6 having a unique fixed point $\bar{U}$. For every initial point $\bar{X} \in \mathcal{K}(C)$, we have $\mathbf{T}^n(\bar{X}) \dot{\rightarrow} \bar{U}$ and $\|\bar{U} - \mathbf{T}^n(\bar{X})\|_{L^1(\delta)} \rightarrow 0$.*

The next simple example illustrates why uniqueness is formulated for the cone $\mathcal{K}(C)$. Even in a deterministic economy, the recursive utility equation may admit several positive solutions, although only one of them belongs to $\mathcal{K}(C)$.

Assume there is no uncertainty and that consumption is constant, $C_t := 1$. Consider the linear certainty equivalent $\mathcal{M}_t(X) = X_{t+1}$. The recursive equation is

$$U_t = \left[(1 - \beta) + \beta U_{t+1}^r \right]^{1/r}.$$

Clearly, the constant process $U_t := \mathbf{1} \in \mathcal{K}(\mathbf{1})$ is a solution of the recursive equation. If we now, for every $A \geq 0$, define

$$U_t^A := \left[1 + A\beta^{-t} \right]^{1/r},$$

then

$$\mathbf{T}(U^A)_t = \left[(1 - \beta) + \beta (U_{t+1}^A)^r \right]^{1/r} = U_t^A$$

Hence U^A is a fixed point of $\mathbf{T}$ for every $A \geq 0$. If $A > 0$ and $r \in (0, 1)$, then $U_t^A \uparrow \infty$, while $U_t^A \downarrow 0$ if $r < 0$. In both cases $U^A \notin \mathcal{K}(C)$ while the only fixed point in $\mathcal{K}(C)$ is $U_t \equiv 1$.

3.5 Non-existence

Propositions 3 and 5 state sufficient conditions for the existence of recursive utilities. We show that those conditions are “almost” necessary, under slightly different hypotheses on the certainty equivalent operator.

Proposition 13. *Let $r \in (0, 1)$ and $\mathcal{M}$ be monotone and positively homogeneous. If*

$$\frac{\mathcal{M}(C)}{C} \geq \beta^{-1/r}, \quad (17)$$

then $\mathbf{T}$ has no fixed point in L_+ .

Thus, when the EIS is above one, adjusted consumption growth cannot be uniformly too high. If the certainty-equivalent value of future consumption is everywhere large enough relative to current consumption, there is no recursive utility.

A perfectly analogous result is valid for $r < 0$, by replacing (17) with the condition

$$\frac{\mathcal{M}(C)}{C} \leq \beta^{1/|r|}, \quad (18)$$

which implies there is no fixed point in L_{++} .

The severe restrictions (17) and (18) guarantee the everywhere non-existence of fixed points. We next provide a weaker result which rules out fixed points in the cone $\mathcal{K}(C)$ only.

Proposition 14. *Let $\mathcal{M}$ be monotone and subhomogeneous. For $r \in (0, 1)$, under the condition*

$$\sup \frac{\mathcal{M}(C)}{C} = \infty$$

no fixed point of $\mathbf{T}$ lies in $\mathcal{K}(C)$.

For $r < 0$ the same result holds whenever

$$\inf \frac{\mathcal{M}(C)}{C} = 0.$$

3.6 Epstein–Zin preferences and related literature

As discussed in the introduction, much of the literature related to ours on recursive utility with unbounded consumption or unbounded state variables focuses on Epstein–Zin preferences. We now make the connection to our framework more explicit. In this case, the certainty equivalent is $\mathcal{M}_t(X) = (\mathbb{E}_t X_{t+1}^s)^{1/s}$, so the recursive utility satisfies

$$U_t = \left[(1 - \beta)C_t^r + \beta \left(\mathbb{E}_t U_{t+1}^s \right)^{r/s} \right]^{1/r}. \quad (19)$$

In terms of the economically relevant parameters $\text{EIS} = (1-r)^{-1}$ and $\text{RRA} = 1-s$, the exponent ratio above is $r/s = (1 - \text{EIS}^{-1})/(1 - \text{RRA})$. The standard expected utility model corresponds to $\text{RRA} = \text{EIS}^{-1}$, while $\text{RRA} > \text{EIS}^{-1}$ corresponds to a preference for early resolution of uncertainty. Markov specifications are discussed more extensively in the next section.

We have already remarked that the operator $\mathbf{T}$ associated with (19) is concave for $s \leq 1$. Setting $g_{t+1} := \log(C_{t+1}/C_t)$, we obtain

$$\frac{\mathcal{M}_t(C)}{C_t} = \mathcal{M}_t\left(\frac{C_{t+1}}{C_t}\right) = \mathcal{M}_t(e^{g_{t+1}}) = (\mathbb{E}_t e^{sg_{t+1}})^{1/s}.$$

Hence, in the Epstein–Zin case, the abstract conditions in Propositions 4 and 6 become restrictions on the conditional certainty-equivalent growth rate of consumption. This is the same object that appears, in transformed form, in the existing literature. The main difference is that those papers often work with transformations of (19), whereas our results are stated directly in terms of the order properties of $\mathbf{T}$.

For instance, setting $u_t = U_t/C_t$ gives

$$u_t = \left[(1 - \beta) + \beta \left(\mathbb{E}_t \left[u_{t+1}^s e^{sg_{t+1}} \right] \right)^{r/s} \right]^{1/r} = \Gamma_r \left(\left(\mathbb{E}_t \left[u_{t+1}^s e^{sg_{t+1}} \right] \right)^{1/s} \right), \quad (20)$$

where $\Gamma_r(t) = [(1 - \beta) + \beta t^r]^{1/r}$.

If U solves (19) in $\mathcal{K}(C)$, then u solves (20) in the cone $\mathcal{K}(\mathbf{1})$. The operator associated with (20) inherits the relevant order properties of $\mathbf{T}$ and will be useful below; in fact, it is closely related to the operator $\mathbf{A}$ of the next section.

Alternatively, we can make a further change of variable by setting

$$\psi_t = u_t^r = \left(\frac{U_t}{C_t} \right)^r.$$

The variable $\psi_t/(1 - \beta)$ is known in the literature as the *wealth-consumption ratio* (see [16]). This gives

$$\psi_t = (1 - \beta) + \beta \left(\mathbb{E}_t \left[\psi_{t+1}^{s/r} e^{sg_{t+1}} \right] \right)^{r/s}.$$

This formulation is used, in different forms, by Hansen and Scheinkman [21] and Pohl, Schmedders, and Wilms [36]. It is especially convenient because it can also handle parameter regions not directly covered by the concavity argument above.

For example, let $r \in (0, 1)$ and $s > r$. The associated operator

$$\mathbf{S}(\psi) = (1 - \beta) + \beta \left(\mathbb{E}_t \left[\psi^{s/r} e^{sg_{t+1}} \right] \right)^{r/s}$$

is monotone and convex. Set

$$M = \sup_{t, \omega} (\mathbb{E}_t e^{sg_{t+1}})^{r/s}.$$

If $M < \beta^{-1}$, then $\mathbf{S}$ maps any interval $[(1 - \beta), \gamma]$ into itself whenever

$$\gamma \geq \bar{\gamma} := \frac{1 - \beta}{1 - \beta M}.$$

Moreover, for every $\gamma > \bar{\gamma}$,

$$\gamma - \mathbf{S}(\gamma) \geq \gamma[1 - \beta M] - (1 - \beta) = (1 - \beta) \left(\frac{\gamma}{\bar{\gamma}} - 1 \right) > 0.$$

Thus no fixed point lies on the *upper perimeter* of the interval $[(1 - \beta), \gamma]$. By the dual formulation of Theorem 1 of [31], it follows that the wealth-consumption recursion has a unique solution in $\mathcal{K}(\mathbf{1})$ for $r \in (0, 1)$ and $s > r$.

Another equivalent transformation is used by Borovička and Stachurski [10, Section II]. Setting $\theta_t = (U_t/C_t)^s$ yields

$$\theta_t = \left[(1 - \beta) + \beta (\mathbb{E}_t [\theta_{t+1} e^{sg_{t+1}}])^{r/s} \right]^{s/r},$$

which can be treated in the same way as the formulations above.

If now we consider the unit-EIS case, the Epstein–Zin recursion becomes

$$U_t = C_t^{1-\beta} \left(\mathbb{E}_t U_{t+1}^s \right)^{\beta/s}.$$

Defining $v_t = s \log (U_t/C_t)$, we have the formulation that, under a convenient restriction, is carefully studied by Christensen [14]:

$$v_t = \beta \log \mathbb{E}_t [\exp (v_{t+1} + sg_{t+1})] \quad (21)$$

as discussed further ahead.

4 Application to macroeconomics

We now specialize the abstract framework developed above to standard macro-finance environments in which recursive utility is commonly applied. We begin with a general Markov specification and then consider four applications. These examples show how the general existence and uniqueness conditions translate into explicit parameter restrictions.

So far, we have formulated intertemporal stochastic models in terms of an abstract probability space $(\Omega, \Sigma, \mathbb{P})$ with information given by an increasing filtration $(\Sigma_t)_{t \in T}$. In the following models the uncertainty is represented by a Markov process. It is thus convenient to provide a brief account of the notation adopted in such a setting⁶.

⁶An excellent description of the present subject can be found in [38]. For a more complete mathematical analysis we refer to [34, Ch. V].

Let $\mathcal{X}$ be a Borel set of a Polish space and set $\Omega = \mathcal{X}^\infty$. The elements of $\mathcal{X}^\infty$ are denoted by $\omega \equiv \xi^\infty = (\xi_t)_{t \geq 0}$. Denote by $\pi_t(\omega) = \xi_t$ the canonical projections for $t \geq 0$ and endow $\mathcal{X}^\infty$ with the canonical filtration $(\Sigma_t)_{t \in T}$ where Σ_t is the least σ -algebra generated by the projections $(\pi_s)_{0 \leq s \leq t}$. The time t -history is given by $\xi^t = (\xi_0, \xi_1, \dots, \xi_t)$. Therefore an adapted Markov process X is a function $X(\omega, t) \equiv X_t(\xi^\infty) \equiv X(\xi^t)$ for $t \in T$.

The probability $\mathbb{P}$ over Σ is generated by an initial distribution π_0 on $\mathcal{X}$ and the stationary transition kernel $Q(\xi, A)$ with the interpretation:

$$Q(\xi_t, A) = \mathbb{P}\{\xi_{t+1} \in A \mid \xi_t\}$$

for $A \in \mathcal{B}(\mathcal{X})$. In this setting, the recursive utility (2) can be written as

$$U(\xi^t) = \left[(1 - \beta)C^r(\xi^t) + \beta(\mathcal{M}_t(U(\xi^t, \cdot)))^r \right]^{1/r}$$

for all $t \geq 0$ and $\xi^{t+1} \in \mathcal{X}^{t+1}$. The associated operator $\mathbf{T}$ is

$$(\mathbf{T}X)(\xi^t) = W(C(\xi^t), \mathcal{M}_t(X(\xi^t, \cdot)))$$

It is convenient in this setting to restrict slightly the functional framework. Unlike what was postulated in the previous sections, here the measurable functions are functions and not equivalence classes.⁷ This will be emphasized by using the calligraphic letters for the spaces involved. For instance, $\mathcal{L}_+$ is the cone of the nonnegative, finitely valued and adapted processes.

The analogue of the space $L^1(\delta)$ will be $\mathcal{L}^1(\delta) \subset \mathcal{L}$. A process X belongs to $\mathcal{L}^1(\delta)$ if all the functions X_t are integrable, i.e., if

$$\|X_t\| = \int |X(\xi^t)| d\mathbb{P} < \infty$$

and

$$\|X\| = \sum_{t \in T} \|X_t\| \delta^t < \infty$$

Endowed with the point-wise order, the space $\mathcal{L}^1(\delta)$ is Dedekind σ -complete and its positive cone is regular. This suffices to apply the results of existence and uniqueness stated in the previous section. Note, incidentally, that $\mathcal{L}^1(\delta)$ is not Dedekind complete.

⁷As a matter of fact, we can also use here the space of classes of functions $L^1(\delta)$. But for some later constructions (for instance, the function h and the operator $\mathbf{A}$) we should require the existence of a stationary distribution for the underlying Markov process.

4.1 Epstein–Zin certainty equivalent

Let us formulate the Epstein–Zin preferences, already introduced in subsection 3.6, in a Markovian setting.

The state of the economy is given by pairs $\xi_t = (z_t, y_t)$ with $z_t \in \mathcal{Z}$ and $y_t \in \mathcal{Y}$, where $\mathcal{Z}$ and $\mathcal{Y}$ are Borel sets of Polish spaces, equipped with their Borel σ -algebras $\mathcal{B}(\mathcal{Z})$ and $\mathcal{B}(\mathcal{Y})$ respectively. The variable z represents the exogenous state of the system, while y is the shock due to the technological innovation.

Assume that the process ξ_t is Markov and that the innovation sequence $(y_t)_{t \geq 1}$ is i.i.d. with law ν , independent of $(z_t)_{t \geq 0}$. Therefore, the probability $\mathbb{P}$ over Σ is generated by the stationary transition kernel

$$\tilde{Q}(\xi, A \times B) = Q(z, A) \times \nu(B)$$

with the following interpretation: given $\xi_t = (z_t, y_t)$,

$$Q(z_t, A) \times \nu(B) = \mathbb{P}\{\xi_{t+1} \in A \times B\} = \mathbb{P}\{z_{t+1} \in A\} \times \nu(B)$$

for every $A \in \mathcal{B}(\mathcal{Z})$ and $B \in \mathcal{B}(\mathcal{Y})$.

Assume that consumption growth is given by the following specification (see Hansen and Scheinkman [21] and Borovička and Stachurski [10])

$$g_{t+1} := \log \frac{C_{t+1}}{C_t} = \kappa(z_t, z_{t+1}, y_{t+1}), \quad t \geq 0, \quad (22)$$

where $\kappa: \mathcal{Z} \times \mathcal{Z} \times \mathcal{Y} \rightarrow \mathbb{R}$ is Borel measurable. Given $\xi = (z, y)$ and $\xi' = (z', y')$, we write $\kappa(\xi, \xi')$ for $\kappa(z, z', y')$.

The Kreps–Porteus certainty equivalent will be

$$\begin{aligned} \mathcal{M}_t(X(\xi^t, \xi_{t+1})) &:= \left(\mathbb{E}^Q[X^s(\xi^t, \xi_{t+1}) \mid \xi^t] \right)^{1/s} \\ &= \left[\int_{\mathcal{Z}} \int_{\mathcal{Y}} X^s(\xi^t, z', y') \nu(dy') Q(z_t, dz') \right]^{1/s}. \end{aligned} \quad (23)$$

Given a consumption process $C \in \mathcal{L}_{++}$, the operator $\mathbf{T} \equiv \mathbf{T}_C$ is

$$(\mathbf{T}(X))_t = W(C_t, \mathcal{M}_t(X_{t+1}))$$

for all $t \in T$ and all infinite histories $\xi^\infty = (\xi_0, \xi_1, \dots)$, whenever $\mathcal{M}(X) \in \mathcal{L}_+$. The Epstein–Zin recursive utility is a fixed point U of $\mathbf{T}$, i.e., $U = (U_t)_{t \in T}$ solves

$$U(\xi^t) = \left[(1 - \beta) C^r(\xi^t) + \beta \left[\int_{\mathcal{Z}} \int_{\mathcal{Y}} U^s(\xi^t, z', y') \nu(dy') Q(z_t, dz') \right]^{r/s} \right]^{1/r}$$

for all $t \in T$ and all ξ^∞ .

Before proceeding in the analysis, let us introduce a more general class of certainty equivalents.

4.2 Positively homogeneous and stationary certainty equivalents

In the Epstein–Zin case, the certainty equivalents are positively homogeneous, i.e., $\mathcal{M}_t(\lambda X_{t+1}) = \lambda \mathcal{M}_t(X_{t+1})$ for all scalars $\lambda > 0$. In fact, the following stronger property, which we call *conditional positive homogeneity*, holds. That is,

$$\mathcal{M}_t(\lambda_t X_{t+1}) = \lambda_t \mathcal{M}_t(X_{t+1}) \quad \forall t \in T,$$

for every admissible $X \in \mathcal{L}_{++}$ and every strictly positive Σ_t -measurable function λ_t .

Another property exhibited by the above certainty equivalents is stationarity. The certainty equivalent $\mathcal{M}$ is called *stationary Markov* if for every admissible nonnegative Borel function $F : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ and for every $s, t \geq 0$, we have

$$\pi_t(\xi^\infty) = \pi_s(\tilde{\xi}^\infty) \implies \mathcal{M}_t(F(\xi_t, \cdot)) = \mathcal{M}_s(F(\tilde{\xi}_s, \cdot))$$

Thus, the evaluation of a one-period Markov payoff depends on neither the calendar date nor the preceding history once the current state is fixed.

Under these two additional assumptions, together with the growth condition (22), a particularly important result can be formulated.

To see this, observe that if $\log(C_{t+1}/C_t) = \kappa(\xi_t, \xi_{t+1})$, then there exists a function $h : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\mathcal{M}_t(C_{t+1}) = C_t h(\xi_t). \tag{24}$$

Indeed, conditional positive homogeneity gives

$$\mathcal{M}_t(C_{t+1}) = C_t \mathcal{M}_t(e^{\kappa(\xi_t, \cdot)}),$$

while stationarity implies that $\mathcal{M}_t(e^{\kappa(\xi, \cdot)})$ depends on the current state ξ but not on calendar time. Hence, defining

$$h(\xi) := \mathcal{M}_t(e^{\kappa(\xi, \cdot)}),$$

yields (24).

For instance, in the Epstein–Zin case the function h is given by

$$h(z) := \left\{ \int_{\mathcal{Z}} \left[\int_{\mathcal{Y}} e^{s\kappa(z, z', y')} \nu(dy') \right] Q(z, dz') \right\}^{1/s}. \tag{25}$$

Consequently, using (24) we have

$$\mathcal{M}(C) \in \mathcal{L}_+ \iff h(z) < \infty \quad \text{for every } z \in \mathcal{Z}. \tag{26}$$

Suppose now that $C \in \mathcal{L}_{++}^1(\delta)$ for some $\delta > 0$, that (26) holds, and $s \in (-\infty, 0) \cup (0, 1]$. Since the Epstein–Zin certainty equivalent is order concave, Propositions 4 and 6 imply uniqueness of recursive utility in $\mathcal{K}(C)$ under the following conditions:

- for $r < 0$:

$$\inf_{z \in \mathcal{Z}} h(z) > \beta^{1/|r|}.$$

- for $r \in (0, 1)$:

$$\sup_{z \in \mathcal{Z}} h(z) < \beta^{-1/r} \quad \text{and} \quad \inf_{z \in \mathcal{Z}} h(z) > 0.$$

Notice that the condition $\inf_{z \in \mathcal{Z}} h(z) > 0$ is the “Markovian” translation of the condition $\mathcal{M}(C)/C \gg 0$ postulated in Proposition 6.

The two properties of homogeneity and stationarity give rise to another significant property regarding the structure of the fixed points.

We have seen that the utilities U_t in the Epstein–Zin equation (19) admit a multiplicative representation $U_t = u_t C_t$ where u_t solves (20). This representation turns out to be true in general under the assumptions of conditional homogeneity and stationarity of the operator $\mathcal{M}$. In fact, from the equation

$$U(\xi^t) = W(C(\xi^t), \mathcal{M}_t(U(\xi^t, \cdot)))$$

it follows

$$\frac{U(\xi^t)}{C(\xi^t)} = W\left(1, \mathcal{M}_t\left(\frac{U(\xi^t, \cdot)}{C(\xi^t)}\right)\right) = W\left(1, \mathcal{M}_t\left(\frac{U(\xi^t, \cdot)}{C(\xi^t, \cdot)} e^{\kappa(\xi_t, \cdot)}\right)\right)$$

Consequently, the substitution $U(\xi^t) = u(\xi^t) C(\xi^t)$ leads to the recursive equation

$$\begin{aligned} u(\xi^t) &= \Gamma_r\left(\mathcal{M}_t\left(u(\xi^t, \cdot) e^{\kappa(\xi_t, \cdot)}\right)\right) \\ \Gamma_r(t) &= ((1 - \beta) + \beta t^r)^{1/r} \end{aligned} \tag{27}$$

We conjecture that, by virtue of the Markovian structure, the utility U takes the *separable* form

$$U(\xi^t) = \varphi(\xi_t) C(\xi^t) \tag{28}$$

Namely, $u(\xi^t) = \varphi(\xi_t)$ for a certain function $\varphi : \mathcal{X} \rightarrow \mathbb{R}$. Said differently, in view of (27), φ solves

$$\varphi(\xi_t) = \Gamma_r\left(\mathcal{M}_t\left(\varphi(\cdot) e^{\kappa(\xi_t, \cdot)}\right)\right) \tag{29}$$

To check this guess, let us introduce the space $B(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ of the bounded measurable functions defined on $\mathcal{X}$. Equipped with the supnorm $\|\varphi\| = \sup_{\mathcal{X}} |\varphi(\xi)|$, this space is a Dedekind σ -complete Banach lattice.

For the sake of simplicity, set

$$\mathcal{M}_t(e^{\kappa(\xi_t, \cdot)} \varphi(\cdot)) = (\mathcal{H}\varphi)(\xi_t).$$

Notice that $\mathcal{H}1 = h$.

In light of equation (29), define the operator

$$\mathbf{A} : B_+(\mathcal{X}, \mathcal{B}(\mathcal{X})) \rightarrow B_+(\mathcal{X}, \mathcal{B}(\mathcal{X}))$$

by

$$(\mathbf{A}\varphi)(\xi) = \Gamma_r(\mathcal{H}\varphi(\xi))$$

where

$$\Gamma_r(t) = ((1 - \beta) + \beta t^r)^{1/r}$$

In other words, we assume that $\mathcal{H}\varphi$ depends only on z for every admissible φ , and identify $\mathbf{A}$ with its restriction to functions of z . For instance, in the Epstein–Zin case, the operator $\mathbf{A}$ takes the explicit form

$$(\mathbf{A}\varphi)(z) = \left\{ (1 - \beta) + \beta \left[\int_{\mathcal{Z}} \int_{\mathcal{Y}} e^{s\kappa(z, z', y')} \varphi(z')^s \nu(dy') Q(z, dz') \right]^{r/s} \right\}^{1/r}.$$

Below we list the main properties of Γ and $\mathbf{A}$ that can be easily checked.

- Γ is increasing for all $r \in \mathbb{R}$ and concave for all $r \leq 1$
- $\mathbf{A}$ is concave and increasing, provided $\mathcal{M}$ is concave and increasing
- $\mathbf{A}$ is σ -order continuous, if $\mathcal{M}$ is σ -order continuous
- If $r \in (0, 1)$, then $\mathbf{A}\varphi \geq (1 - \beta)^{1/r}$ for all $\varphi \geq 0$. If $r < 0$, then $0 \leq \mathbf{A}\varphi \leq (1 - \beta)^{1/r}$.

The next fixed-point result holds under assumptions (i)–(iii), together with conditional homogeneity and stationarity. It follows directly from the properties listed above and the existence and uniqueness arguments of Section 3.

Proposition 15. *Let $C \in \mathcal{L}_{++}^1(\delta)$ for some $\delta > 0$. Under assumptions (i)–(iii), conditional homogeneity and stationarity:*

1. *for $0 < r < 1$, set $\bar{h} := \sup_{z \in \mathcal{Z}} h(z)$. Under the condition*

$$\bar{h} < \beta^{-1/r}, \tag{30}$$

$\mathbf{A}$ maps the interval $[(1 - \beta)^{1/r}, b]$ into itself and has a least and a greatest fixed point there, for every

$$b \geq \bar{b} := \left(\frac{1 - \beta}{1 - \beta \bar{h}^r} \right)^{1/r},$$

In addition, if $\mathcal{M}$ is order concave, then the fixed point is unique in $B_{++}(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.

2. for $r < 0$, set $\underline{h} := \inf_{z \in \mathcal{Z}} h(z)$ and suppose $h(z) < \infty$ for every $z \in \mathcal{Z}$. Under

$$\underline{h} > \beta^{1/|r|},$$

$\mathbf{A}$ maps $[b, (1 - \beta)^{1/r}]$ into itself and has a least and a greatest fixed point there, for every

$$0 < b \leq \underline{b} := \left(\frac{1 - \beta}{1 - \beta \underline{h}^r} \right)^{1/r},$$

If $\mathcal{M}$ is order concave, the fixed point is unique in $\mathcal{K}(1)$.

Unlike the uniqueness result for $\mathbf{T}$ in Section 3, uniqueness of the fixed point of $\mathbf{A}$ does not require the condition $\inf_{z \in \mathcal{Z}} h(z) > 0$. Thus, even when $\inf_{z \in \mathcal{Z}} h(z) = 0$, $\mathbf{A}$ may have a unique fixed point while $\mathbf{T}$ may admit multiple fixed points. Hence, with non-unit EIS, separability provides a natural selection criterion: the separable recursive utility is unique in $\mathcal{K}(C)$ even when recursive utility more generally need not be.

Remark. Let us outline an alternative route to handle the operator $\mathbf{A}$. Suppose, for simplicity, that $r \in (0, 1)$ and $\mathbf{A}$ is concave. The lower bound $\mathbf{A}\varphi \geq (1 - \beta)^{1/r}$ implies that

$$\mathbf{A}(\alpha\varphi) = \mathbf{A}(\alpha\varphi + \bar{\alpha}0) \geq \alpha\mathbf{A}(\varphi) + \bar{\alpha}\mathbf{A}(0) \geq \alpha\mathbf{A}(\varphi) + \bar{\alpha}(1 - \beta)^{1/r}$$

The property $\mathbf{A}(\alpha\varphi) \geq \alpha\mathbf{A}(\varphi) + \bar{\alpha}k$, for every $\alpha \in (0, 1)$ and for some $k > 0$, is called *k-subconcavity* (at 0); see [31]. This in turn implies that the self-map $\mathbf{A} : [0, b] \rightarrow [0, b]$ for any given $b \geq \bar{b}$ is *p-subhomogeneous* for some $p \in (0, 1)$; see Definition 7 and [31, Proposition 6]. Namely,

$$\mathbf{A}(\alpha\varphi) \geq \alpha^p \mathbf{A}(\varphi) \text{ for } \alpha \in (0, 1)$$

If we introduce the Thompson distance d_T (see [39] and [31]) in the cone $\mathcal{K}(1)$, the map $\mathbf{A} : [0, b] \rightarrow [0, b]$ turns out to be a *p-contraction*, i.e. $d_T(\mathbf{A}\varphi_1, \mathbf{A}\varphi_2) \leq p d_T(\varphi_1, \varphi_2)$. One can then deduce the existence of a unique fixed point for the map $\mathbf{A}$ and also the convergence of the iterates in the norm $\|\cdot\|$.

Proposition 15 shows that our conjecture about the multiplicative relation (28) is true. This is formalized in the next proposition, whose trivial proof is omitted.

Proposition 16. Let $0 \neq r < 1$, $C \in \mathcal{L}_{++}^1(\delta)$ and $\bar{\varphi}$ be a fixed point of $\mathbf{A}$. Then the process $U \in \mathcal{L}_{++}^1(\delta)$ defined by $U_t = \bar{\varphi}(z_t)C_t$ for every $t \geq 0$ is a fixed point of $\mathbf{T}$.

Notice that the previous result does not require the fixed point $\varphi = \mathbf{A}\varphi$ to belong to $\mathcal{K}(1)$, nor does it require $\mathcal{M}$ to be concave, for $U_t = \varphi(z_t)C_t$ to be a fixed point of $\mathbf{T}$. Clearly, $U \in \mathcal{K}(C)$ if and only if $\varphi \in \mathcal{K}(1)$.

4.3 Cobb–Douglas aggregator

Finally, let us discuss the case of the Cobb–Douglas aggregator in the Markov setting. It is now convenient to assume $\alpha + \beta = 1 \iff p = 1$. We add the hypotheses of conditional positive homogeneity and stationarity.

In view of Proposition 9, if $\alpha + \beta = 1$ then there is a unique solution in $\mathcal{K}(C)$, provided $C \in L_{++}^1(\delta)$ and $h \in \mathcal{K}(1)$ where $h : \mathcal{Z} \rightarrow \mathbb{R}$ is given by $h(\xi_t) = \mathcal{M}_t(e^{\kappa(\xi_t, \cdot)})$.

As we did earlier for the CES aggregator, we associate with $\mathbf{T}$ an operator $\mathbf{A}$. In fact, if the recursive utility U is a solution to

$$U(\xi^t) = C^\alpha(\xi^t) [\mathcal{M}_t(U(\xi^t, \cdot))]^\beta \quad (31)$$

the substitution $u(\xi^t) = U(\xi^t)/C(\xi^t)$ yields the recursion

$$u(\xi^t) = \mathcal{M}_t^\beta [u(\xi^t, \cdot) e^{\kappa(\xi_t, \cdot)}]$$

Hence the operator $\mathbf{A}$ will take the form

$$\mathbf{A}\varphi = \mathcal{M}_t^\beta [\varphi(\cdot) e^{\kappa(\xi_t, \cdot)}]$$

For instance, in the Epstein–Zin case it is

$$(\mathbf{A}\varphi)(z) = \left(\int_{\mathcal{Z}} \int_{\mathcal{Y}} e^{s\kappa(z, z', y')} \varphi(z')^s \nu(dy') Q(z, dz') \right)^{\beta/s}.$$

Clearly the operator $\mathbf{A}$ turns out to be β -subhomogeneous. Existence and uniqueness are then assured in the cone $\mathcal{K}(1)$, provided $\varepsilon \leq h(z) \leq M$ for all $z \in \mathcal{Z}$.

In the Epstein–Zin case we have seen (eq. 21 of Section 3.6) the alternative use of the transformation $v(\xi^t) = s \log(U(\xi^t)/C(\xi^t))$ that gives the recursion

$$v(\xi^t) = \beta \log \mathbb{E}^Q \left[\exp \left(v(\xi^t, \xi_{t+1}) + s\kappa(\xi_t, \xi_{t+1}) \right) \middle| \xi^t \right] \quad (32)$$

Looking for the solutions of the kind $v(\xi^t) \equiv \varphi(\xi_t)$, we get the recursive equation

$$\varphi(\xi) = \beta \log \mathbb{E}^Q [\exp(\varphi(\xi_{t+1}) + s\kappa(\xi_t, \xi_{t+1})) | \xi_t = \xi] \quad (33)$$

extensively studied by Christensen [14].

To clarify the deep difference between (32) and (33), suppose that $U \in \mathcal{K}(C)$ is a solution to (31). Hence

$$U(\xi^t) = C(\xi^t) \bar{\varphi}(\xi_t)$$

for some $\bar{\varphi} \in \mathcal{K}(1)$. As discussed in subsection 3.3 the utility U generates a family of utilities $V(\xi^t) = \lambda^{\beta^{-t}} C(\xi^t) \bar{\varphi}(\xi_t)$, with $\lambda \in (0, 1]$. Under the change of variables $v(\xi^t) = s \log(V(\xi^t)/C(\xi^t))$ we obtain the family of solutions to the recursion (32):

$$v(\xi^t) = s\beta^{-t} \log \lambda + s \log \bar{\varphi}(\xi_t)$$

while the unique bounded measurable solution to (33) is given by $\varphi(\xi) = s \log \bar{\varphi}(\xi)$.

Observe that the condition $h \in \mathcal{K}(\mathbf{1})$, which guarantees existence and uniqueness in $\mathcal{K}(C)$, may fail even though fixed points exist in other cones. Section 4.6 provides an economically relevant example of this phenomenon. Moreover, Section 4.7 establishes multiplicity of separable recursive utilities for the constant-volatility Case I consumption specification of Bansal and Yaron [5] when the EIS is set to one.

4.4 Mehra–Prescott model

We begin by illustrating our results with the simplest finite-state member of the general Markov family described in the previous section (see Mehra and Prescott [33]). The state space is the finite set $\mathcal{Z} = \{1, \dots, m\}$ and the Markov chain is generated by the stochastic matrix $\Pi = (\pi_{ij})_{i,j=1}^m$. The initial distribution is the m -dimensional row vector $\pi_0 = (\pi_0^1, \pi_0^2, \dots, \pi_0^m)$ and

$$Q(i, \{j\}) = \pi_{ij}.$$

For each $i \in \{1, \dots, m\}$, let $\ell_i > 0$ denote the gross consumption growth associated with state i . The consumption growth depends only on the next state. Namely, $C_{t+1} = \ell_{i_{t+1}} C_t$ and

$$g_{t+1} = \kappa(i_t, i_{t+1}) = \log \ell_{i_{t+1}}.$$

Thus the Mehra–Prescott economy is exactly the general Markov model with a finite state space and no auxiliary innovation.

Here, we consider the Epstein–Zin certainty equivalent (23). Therefore, in this case the function $h : \mathcal{Z} \rightarrow \mathbb{R}$ in (25) is given by

$$h(i) = \left(\sum_{j=1}^m \pi_{ij} \ell_j^s \right)^{1/s}, \quad i \in \mathcal{Z}.$$

Let $D := \text{diag}(\ell_1, \dots, \ell_m)$, and let $\mathbf{1} = (1, 1, \dots, 1)$ be the m -dimensional row vector of ones. Tedious algebra leads to the moment condition

$$\mathbb{E}[C_t] = C_0 \left\langle \pi_0 (\Pi D)^t, \mathbf{1} \right\rangle \quad \forall t \geq 0.$$

Hence

$$C \in L_{++}^1(\delta) \iff \sum_{t \geq 0} \delta^t \left\langle \pi_0 (\Pi D)^t, \mathbf{1} \right\rangle < \infty.$$

If $\rho(A)$ denotes the spectral radius of a matrix A , then the condition $\delta \rho(\Pi D) < 1$ suffices to get $C \in L_{++}^1(\delta)$.

Thus, under the assumption $\delta \rho(\Pi D) < 1$, we obtain the following conditions for existence, uniqueness, and non-existence. If $0 < r < 1$, then

$$\max_i h(i) < \beta^{-1/r}$$

implies existence, uniqueness, and global attractivity, whereas $\min_i h(i) \geq \beta^{-1/r}$ implies that there is no fixed point in $\mathcal{L}_+$. If $r < 0$, then

$$\min_i h(i) > \beta^{1/|r|}$$

implies existence, uniqueness, and global attractivity, whereas $\max_i h(i) \leq \beta^{1/|r|}$ implies that there is no fixed point in $\mathcal{L}_{++}$.

We deduce from Proposition 16 that the recursive utility has the separable form $U_t = \varphi(i_t)C_t$, where the vector $\varphi = (\varphi_1, \dots, \varphi_m)$ is the unique solution to the system of non-linear equations

$$\varphi_i = \left[(1 - \beta) + \beta \left(\sum_{j=1}^m \pi_{ij} \ell_j^s \varphi_j^s \right)^{r/s} \right]^{1/r}, \quad i = 1, \dots, m.$$

For instance, in the calibration used by Weil [41] for the Mehra–Prescott model, there are two states, $m = 2$, with gross consumption growth $\ell_1 = 1.054$, $\ell_2 = 0.984$ and transition matrix

$$\Pi = \begin{pmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{pmatrix}, \quad \pi_0 = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}.$$

We use Weil’s calibration with $\beta = 0.95$ and consider $\text{EIS} \in [0.1, 1) \cup (1, \infty)$ and $\text{RRA} \in (0, 1) \cup (1, 10]$, corresponding to $r, s \in [-9, 0) \cup (0, 1)$. Over this parameter range,

$$1.008 \leq h(i) \leq 1.024, \quad i = 1, 2.$$

Moreover, $\beta^{1/|r|} \leq 0.95^{1/9} \simeq 0.9943$ for $r < 0$, while $\beta^{-1/r} > 0.95^{-1} \simeq 1.053$ for $r > 0$. Hence the relevant existence and uniqueness conditions are satisfied for every $r, s \in [-9, 0) \cup (0, 1)$, and there is a unique recursive utility $U \in \mathcal{K}(C)$ throughout this parameter range.

To illustrate what parameters would lead to non-existence, assume that $s = -9$. In the high-EIS case, a common 5% increase in the two growth rates, from $(\ell_1, \ell_2) = (1.054, 0.984)$ to $(1.1067, 1.0332)$, gives $\min_i h(i) \simeq 1.059$. Hence the non-existence condition holds whenever $\min_i h(i) \geq 0.95^{-1/r}$, that is, for

$$0.8964 \simeq -\frac{\log(0.95)}{\log(\min_i h(i))} \leq r < 1.$$

In the low-EIS case, a common 3% reduction, from $(1.054, 0.984)$ to $(1.02238, 0.95448)$, gives $\max_i h(i) \simeq 0.9874$. Hence the low-EIS non-existence condition holds whenever $\max_i h(i) \leq 0.95^{1/|r|}$, that is, for $-9 \leq r \leq -\frac{\log(0.95)}{\log(\max_i h(i))} \simeq -4.041$, over the parameter range of interest.

Finally, the unique fixed point can be found by solving the system of equations

$$\begin{aligned}\varphi_1 &= \left[0.05 + 0.95 (0.43(1.054\varphi_1)^s + 0.57(0.984\varphi_2)^s)^{r/s}\right]^{1/r}, \\ \varphi_2 &= \left[0.05 + 0.95 (0.57(1.054\varphi_1)^s + 0.43(0.984\varphi_2)^s)^{r/s}\right]^{1/r}.\end{aligned}$$

For example, when $r = -1$ and $s = -9$, corresponding to $\text{RRA} = 10$ and $\text{EIS} = 1/2$, the unique solution is

$$\varphi_1 \simeq 1.263, \quad \varphi_2 \simeq 1.273,$$

so that

$$U_t \simeq \begin{cases} 1.263 C_t, & \text{if } i_t = 1, \\ 1.273 C_t, & \text{if } i_t = 2. \end{cases}$$

To be clear, here and below, the approximation sign refers only to the reported numerical values of the coefficients.

4.5 Hidden-regime consumption growth

We now specialize the general Markov consumption-growth model of Section 4.1 to a two-state hidden-regime learning economy. Models of this kind are widely used in macro-finance because they provide a parsimonious way to represent persistent changes in expected growth and macroeconomic risk. Here we focus on the model considered by Ju and Miao [23] (see also Lettau, Ludvigson and Wachter [28]). For this model, Christensen [14] establishes existence and uniqueness with unit EIS, while Pohl et al. [36] establish existence with non-unit EIS; uniqueness in the latter case had remained unresolved.

Consumption growth is determined by a regime that is not observed by the agent. If the economy is in regime $i \in \{1, 2\}$, then consumption growth is given by

$$g_{t+1} = \log \frac{C_{t+1}}{C_t} = \bar{g}_i + \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, 1), \quad i = 1, 2,$$

where $\bar{g}_1 > \bar{g}_2$ and $\sigma > 0$. Thus, 1 is the high-growth regime, while regime 2 is the low-growth regime. Here, the shocks (ε_t) are i.i.d., normal and independent of the regime process. The regime process follows a Markov chain with transition matrix

$$\Lambda = \begin{pmatrix} \lambda_{11} & 1 - \lambda_{11} \\ \lambda_{21} & 1 - \lambda_{21} \end{pmatrix},$$

where λ_{11} is the probability of remaining in regime 1 and λ_{21} is the probability of moving from regime 2 to regime 1; see equation (13) in [23]. Observe that C_t has unbounded support for every $t \geq 1$.

Because the agent observes consumption growth but not the underlying regime, the relevant state variable is the agent's belief μ_t that the next-period regime is the

high-growth regime. In particular, the hidden-regime model of Ju and Miao [23] can be recast into our framework as a Markov economy with state μ_t , as we formally show in Supplemental Appendix S.2. In the following, we work directly with our Markov formulation.

In particular, we can take the state space to be $\mathcal{Z} = [\lambda_{21}, \lambda_{11}] := K$ and, for every $\mu' \in (\lambda_{21}, \lambda_{11})$, the function κ in (22) is given by

$$\kappa(\mu, \mu') := \frac{\bar{g}_1 + \bar{g}_2}{2} + \frac{\sigma^2}{\bar{g}_1 - \bar{g}_2} \left[\log \frac{\mu' - \lambda_{21}}{\lambda_{11} - \mu'} - \log \frac{\mu}{1 - \mu} \right]. \quad (34)$$

For $i = 1, 2$, let f_i denote the density of $N(\bar{g}_i, \sigma^2)$, and define

$$J(\mu, \mu') := \frac{\sigma^2(\lambda_{11} - \lambda_{21})}{(\bar{g}_1 - \bar{g}_2)(\mu' - \lambda_{21})(\lambda_{11} - \mu')}.$$

For every Borel set $A \subseteq K$, define Q by

$$Q(\mu, A) = \int_A f_\mu(\kappa(\mu, \mu')) J(\mu, \mu') d\mu',$$

where

$$f_\mu(g) := \mu f_1(g) + (1 - \mu) f_2(g).$$

For $i = 1, 2$, define the regime- i conditional kernel by

$$Q_i(\mu, d\mu') := \frac{f_i(\kappa(\mu, \mu'))}{f_\mu(\kappa(\mu, \mu'))} Q(\mu, d\mu').$$

Thus, $Q_i(\mu, \cdot)$ is the conditional distribution of μ_{t+1} given that $\mu_t = \mu$ and that the economy is in regime i in period $t + 1$.

Finally, as we show in the Supplemental Appendix S.2, a sufficient condition for $C \in \mathcal{L}_{++}^1(\delta)$ is

$$\delta \exp\left(\frac{1}{2}\sigma^2\right) \left[\lambda_{11} e^{\bar{g}_1} + (1 - \lambda_{11}) e^{\bar{g}_2} \right] < 1. \quad (35)$$

We are going to study the recursive utility in this model under three distinct preference specifications: smooth ambiguity, Hansen–Sargent robust control, and multiple priors. We focus on the calibration considered by Ju and Miao [23].

Smooth ambiguity and preferences. Smooth ambiguity is a popular model of ambiguity (e.g., see [25, 26, 22]). We follow the specification in Ju and Miao [23, eq. (8)]. For $i = 1, 2$, define the certainty equivalent conditional on regime i by

$$\mathcal{M}_t^i(C_{t+1}) = \left[\int_K \left(C_t e^{\kappa(\mu, \mu')} \right)^s Q_i(\mu, d\mu') \right]^{1/s}.$$

The smooth-ambiguity certainty equivalent is then

$$\mathcal{M}_t(C_{t+1}) = \left\{ \mu_t \left[\mathcal{M}_t^1(C_{t+1}) \right]^\eta + (1 - \mu_t) \left[\mathcal{M}_t^2(C_{t+1}) \right]^\eta \right\}^{1/\eta}.$$

Thus, the evaluation proceeds in two stages. First, risk is evaluated conditional on each possible regime, with $\text{RRA} = 1 - s$ reflecting risk aversion. These regime-specific certainty equivalents are then combined using the current probabilities μ_t and $1 - \mu_t$, with $1 - \eta$ reflecting the sensitivity to ambiguity about which regime will prevail. In particular, the case $\eta = s$ reduces to the Epstein–Zin certainty equivalent. It is immediate to verify that the smooth-ambiguity certainty equivalent is monotone, positively homogeneous, stationary Markov, σ -order continuous, and when $0 \neq \eta, s \leq 1$ it is also order concave.

For this certainty equivalent (see Appendix S.2.1) the function $h : K \rightarrow \mathbb{R}$ can be computed in closed form and is given by

$$h(\mu) = \left\{ \mu \exp \left[\eta \left(\bar{g}_1 + \frac{1}{2} s \sigma^2 \right) \right] + (1 - \mu) \exp \left[\eta \left(\bar{g}_2 + \frac{1}{2} s \sigma^2 \right) \right] \right\}^{1/\eta}.$$

The calibration of Ju and Miao [23] is summarized in Table 1. Under this calibration, the belief support is $K = [\lambda_{21}, \lambda_{11}] = [0.484, 0.978]$.

Parameter	Value	Description
λ_{11}	0.978	Persistence of the high-growth regime
λ_{22}	0.516	Persistence of the low-growth regime
$\bar{g}_1$	0.02251	Mean growth in the high-growth regime
$\bar{g}_2$	−0.06785	Mean growth in the low-growth regime
σ	0.03127	Conditional standard deviation of growth
β	0.975	Continuation weight
r (EIS)	1/3 (3/2)	CES parameter (EIS)
s (RRA)	−1 (2)	Risk aversion power (RRA)
η ($1 - \eta$)	−7.864 (8.864)	Ambiguity aversion power (ambiguity aversion)

Table 1: Ju and Miao calibration.

For these parameter values,

$$\exp \left(\frac{1}{2} \sigma^2 \right) \left[0.978 e^{\bar{g}_1} + (1 - 0.978) e^{\bar{g}_2} \right] \simeq 1.021.$$

Therefore, from (35), we obtain $C \in \mathcal{L}_{++}^1(\delta)$ whenever $0 < \delta < 1.021^{-1} \simeq 0.9794$.

Let us verify the existence and uniqueness conditions. The function h is continuous and strictly positive on the compact set K . Consequently, it is bounded above and bounded away from zero, so the existence condition also implies the additional lower bound required for uniqueness. Since $r = 1/3 > 0$, the relevant existence condition is

$$h(\mu) < \beta^{-1/r} = 0.975^{-3} \simeq 1.079 \quad \text{for every } \mu \in K.$$

As shown in Supplemental Appendix S.2.1, h is increasing, and hence

$$h(\mu) \leq h(0.978) \simeq 1.019 < \beta^{-1/r},$$

so that the existence condition is satisfied. Hence the smooth-ambiguity recursion has a unique globally attractive fixed point in $\mathcal{K}(C)$.

The non-existence condition for $0 < r < 1$ is

$$h(\mu) \geq \beta^{-1/r} \simeq 1.079 \quad \text{for every } \mu \in K.$$

It can be satisfied with higher mean growth and a less persistent low-growth regime. For instance, suppose that $\lambda_{21} = 0.800$, and hence $\lambda_{22} = 0.200$, thereby lowering the probability of remaining in the low-growth regime from 0.516 to 0.200. Raising both mean log-growth rates from $(\bar{g}_1, \bar{g}_2) = (0.02251, -0.06785)$ to $(0.10251, 0.01215)$ increases gross growth in each regime by about 8.3%, in which case we obtain

$$\min_{\mu \in K} h(\mu) \simeq 1.081 > \beta^{-1/r} \simeq 1.079,$$

implying that with this change in parameters there is no recursive utility in $\mathcal{L}_+$.

Finally, by Proposition 16 the unique fixed point in $\mathcal{K}(C)$ takes the form

$$U_t = \varphi(\mu_t)C_t,$$

where $\varphi : K \rightarrow \mathbb{R}_{++}$ is the unique positive solution of

$$\begin{aligned} \varphi(\mu) = & \left\{ (1 - \beta) + \beta \left[\mu \left(\int_K \left(e^{\kappa(\mu, \mu')} \varphi(\mu') \right)^s Q_1(\mu, d\mu') \right)^{\eta/s} \right. \right. \\ & \left. \left. + (1 - \mu) \left(\int_K \left(e^{\kappa(\mu, \mu')} \varphi(\mu') \right)^s Q_2(\mu, d\mu') \right)^{\eta/s} \right]^{r/\eta} \right\}^{1/r}, \end{aligned}$$

for every $\mu \in K$.

Hansen–Sargent robust-control preferences. Let us study the recursive utility under Hansen–Sargent robust-control preferences (unit EIS), again under the calibration of Ju and Miao [23]. Thus, the only change relative to the previous application is that we set $r = 0$. The CES aggregator is then replaced by $W(x, y) = x^{1-\beta}y^\beta$ and the recursive utility operator is then

$$(\mathbf{T}X)_t = C_t^{1-\beta} [\mathcal{M}_t(X_{t+1})]^\beta.$$

Hence a recursive utility is a fixed point $U = \mathbf{T}U$, or, equivalently by taking logarithms

$$\log U_t = (1 - \beta) \log C_t + \beta \log \mathcal{M}_t(U_{t+1}).$$

This is the unit-EIS recursion associated with Hansen–Sargent robust-control preferences [19]; see also equation (9) in Ju and Miao [23].

By Proposition 9, first note that as before the condition $C \in L_{++}^1(\delta)$ holds whenever $\delta < 0.9794$. Therefore, the condition for existence and uniqueness is $\mathcal{M}(C) \in \mathcal{K}(C)$. We have that

$$\mathcal{M}_t(C_{t+1}) = C_t h(\mu_t),$$

and

$$\min_{\mu} h(\mu) \simeq 0.9681, \quad \max_{\mu} h(\mu) \simeq 1.019.$$

Thus, we infer that $\mathcal{M}(C) \in \mathcal{K}(C)$ is true and so existence and uniqueness hold in $\mathcal{K}(C)$.

Multiple priors. The multiple-priors specification is obtained as the limiting case of smooth ambiguity when ambiguity aversion becomes infinite. In our parametrization, this corresponds to $\eta \rightarrow -\infty$ (see equation (11) in Ju and Miao [23]). We keep all the remaining parameters at the values reported in Table 1. Since the consumption process is unchanged, the sufficient condition for $C \in \mathcal{L}_{++}^1(\delta)$ is unchanged as well.

The multiple-priors certainty equivalent $\mathcal{M}^{\text{MP}}$ is given by

$$\mathcal{M}_t^{\text{MP}}(C_{t+1}) = \min_{i \in \{1,2\}} \mathcal{M}_t^i(C_{t+1}) = \min_{i \in \{1,2\}} \left[\int_K \left(C_t e^{\kappa(\mu_t, \mu_{t+1})} \right)^s Q_i(\mu_t, d\mu') \right]^{1/s}$$

The agent therefore evaluates continuation utility under the least favorable regime. For $0 \neq s \leq 1$, this operator is monotone, positively homogeneous, stationary Markov, σ -order continuous, and order concave.

Since $\bar{g}_1 > \bar{g}_2$, regime 2 is the least favorable regime for next-period consumption. A direct calculation gives

$$\mathcal{M}_t^{\text{MP}}(C_{t+1}) = C_t \min_{i \in \{1,2\}} \exp \left(\bar{g}_i + \frac{1}{2} s \sigma^2 \right) = C_t \exp \left(\bar{g}_2 + \frac{1}{2} s \sigma^2 \right) = C_t h^{\text{MP}},$$

where

$$h^{\text{MP}} := \exp \left(\bar{g}_2 + \frac{1}{2} s \sigma^2 \right).$$

The function h^{MP} does not depend on the state variable. As $s = -1$, $\beta = 0.975$ and $r = 1/3$, we get

$$0 < h^{\text{MP}} = \exp \left(-0.06785 - \frac{1}{2} (0.03127)^2 \right) \simeq 0.9339 < \beta^{-1/r} \simeq 1.079.$$

Hence the sufficient conditions for existence and uniqueness are satisfied, and so there is a unique globally attractive fixed point in $\mathcal{K}(C)$.

Finally, consider a common increase $d > 0$ in the two mean log-growth rates:

$$(\bar{g}_1, \bar{g}_2) \mapsto (\bar{g}_1 + d, \bar{g}_2 + d).$$

By Proposition 13, no recursive utility exists in $\mathcal{L}_+$ whenever

$$e^d h^{\text{MP}} \geq \beta^{-1/r} \iff d \geq \log \left(\frac{\beta^{-1/r}}{h^{\text{MP}}} \right) \simeq 0.1443,$$

which corresponds approximately to a 15.5% increase in both growth rates. Note that, unlike in the smooth-ambiguity case, this threshold does not depend on the persistence of the low-growth regime, since h^{MP} is unaffected by the transition probabilities.

4.6 Epstein–Zin with rare disasters

We next examine a rare-disaster economy with persistent disaster intensity studied by Bidder and Smith [7] (see also Wachter [40]), with the difference that we adopt Epstein–Zin preferences with a non-unit elasticity of intertemporal substitution. This example illustrates the distinction between uniqueness of recursive utility, in the general setting, and uniqueness within the separable class. With non-unit EIS, our existence conditions hold while the conditions guaranteeing uniqueness of recursive utility fail. Nevertheless, our results imply a unique separable recursive utility in $\mathcal{K}(C)$. By contrast, at unit EIS the same environment admits two distinct separable recursive utilities.

In this model, consumption growth satisfies

$$\log \frac{C_{t+1}}{C_t} = \nu_g + \nu_j j_{t+1} + \sigma_j \sqrt{j_{t+1}} \eta_{t+1} + \sigma \varepsilon_{t+1},$$

where j_{t+1} denotes the number of disasters occurring between times t and $t+1$, while η_{t+1} and ε_{t+1} are independent standard normal random variables. The parameter $\nu_j < 0$ is the average effect of one disaster, and because $\nu_j < 0$, each additional disaster lowers expected consumption growth.

In particular, the number of disasters at time $t+1$ is distributed according to a Poisson distribution with parameter H_t , that is, $j_{t+1} \mid H_t \sim \text{Poisson}(H_t)$, independently of η_{t+1} and ε_{t+1} . In words, the number of disasters depends on the Poisson intensity parameter H_t , which is the state variable in this model. We therefore refer to H_t as disaster intensity.

Observe that if we consider an i.i.d. sequence $(u_t)_{t \geq 0}$ such that $u_{t+1} \sim \text{Unif}(0, 1)$ independent of H_t , and define

$$J(H, u) := \min \left\{ j \in \{0, 1, 2, \dots\} : \sum_{n=0}^j e^{-H} \frac{H^n}{n!} \geq u \right\},$$

then the realized number of disasters can be written as $j_{t+1} = J(H_t, u_{t+1})$.⁸ In this case the function κ is given by

$$\kappa(H, H', y) := \nu_g + \nu_j J(H, u) + \sigma_j \sqrt{J(H, u)} \varepsilon_z + \sigma \varepsilon_g,$$

for every $y = (u, \varepsilon_z, \varepsilon_g) \in \mathcal{Y} := (0, 1) \times \mathbb{R} \times \mathbb{R}$.

In particular, the disaster intensity H_t follows a first-order autoregressive gamma process (see Gouriéroux and Jasiak [17, eq. (2.1) and Definition 1]). Therefore, the density of the transition kernel Q is

$$q(H, H') = e^{-\phi H/c} \sum_{n=0}^{\infty} \frac{(\phi H/c)^n}{n!} \frac{(H')^{d_H+n-1} e^{-H'/c}}{c^{d_H+n} \Gamma(d_H + n)}, \quad H, H' \in \mathbb{R}_+, \quad (36)$$

where $d_H > 0$, $c > 0$, $0 \leq \phi < 1$, and $\Gamma(\cdot)$ denotes Euler's gamma function. Supplemental Appendix S.3 verifies that

$$\mathbb{E}[H_{t+1} \mid H_t = H] = cd_H + \phi H.$$

Notice that the parameter ϕ determines the persistence of disaster intensity.

As mentioned, we use the Epstein–Zin certainty equivalent

$$\mathcal{M}_t(X_{t+1}) = \left(\mathbb{E}[X_{t+1}^s \mid H_t] \right)^{1/s}.$$

Consequently the function $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies

$$h(H) := \frac{\mathcal{M}_t(C_{t+1})}{C_t} = \left(\mathbb{E}[e^{sg_{t+1}} \mid H_t = H] \right)^{1/s}. \quad (37)$$

As shown in Supplemental Appendix S.3,

$$h(H) = B_s e^{A_s H},$$

where

$$B_s = \exp\left(\nu_g + \frac{1}{2}s\sigma^2\right), \quad A_s = \frac{\exp\left(s\nu_j + \frac{1}{2}s^2\sigma_j^2\right) - 1}{s}.$$

In the calibration considered below, $A_s < 0$, so $h(H)$ decreases with H : states with greater disaster risk have lower “expected” growth.

Supplemental Appendix S.3 also shows that a sufficient condition for $C \in \mathcal{L}_{++}^1(\delta)$ is

$$\nu_j + \frac{1}{2}\sigma_j^2 \leq 0 \quad \text{and} \quad 0 < \delta < \min\left\{1, \exp\left(-\nu_g - \frac{1}{2}\sigma^2\right)\right\}.$$

We now verify the existence and uniqueness conditions under the standard monthly rare-disaster calibration of Bidder and Smith [7]. The only preference parameter we

Parameter	Value	Description
ν_g	1.50×10^{-3}	Mean non-disaster growth component
σ^2	3.30×10^{-5}	Variance of the Gaussian growth shock
ν_j	-0.30	Mean log effect of one disaster
σ_j	0.10	Standard deviation of an individual disaster effect
ϕ	0.993	Persistence of disaster intensity
d_H	1.788	Shape parameter of the ARG intensity process
c	7.766×10^{-6}	Scale parameter of the ARG intensity process
β	0.9992	Continuation weight
r (EIS)	$9/19$ (19/10)	CES parameter (EIS)
s (RRA)	-2.78 (3.78)	Risk aversion power (RRA)

Table 2: Rare-disaster calibration, with economic counterparts in parentheses.

change is the EIS, which we set equal to 1.9 (equivalently $r = 9/19$). The parameter values are reported in Table 2.

For these parameters,

$$\nu_j + \frac{1}{2}\sigma_j^2 = -0.295 < 0,$$

so that $C \in \mathcal{L}_{++}^1(\delta)$ for every $0 < \delta < \min \{1, e^{-\nu_g - \frac{1}{2}\sigma^2}\} \simeq 0.9985$. Moreover,

$$B_s = \exp\left(\nu_g + \frac{1}{2}s\sigma^2\right) \simeq 1.00146, \quad A_s \simeq -0.5012 < 0.$$

As $r = 9/19 \in (0, 1)$, the relevant existence condition is

$$\sup_{H \geq 0} h(H) < \beta^{-1/r}.$$

Since $A_s < 0$, the function h is decreasing and its supremum is attained at $H = 0$. Therefore,

$$\sup_{H \geq 0} h(H) = h(0) = B_s \simeq 1.00146 < \beta^{-1/r} = 0.9992^{-19/9} \simeq 1.00169.$$

Thus, the existence condition guarantees at least one recursive utility in $\mathcal{K}(C)$.

The uniqueness condition of Proposition 6, however, fails. Since $A_s < 0$, it follows that $\lim_{H \uparrow \infty} h(H) = 0$ and hence

$$\inf_{H \geq 0} h(H) = \inf_C \frac{\mathcal{M}(C)}{C} = 0.$$

⁸To see this, fix $H_t \geq 0$ and let $F_{H_t}(j) := \sum_{n=0}^j e^{-H_t} H_t^n / n!$ be the cumulative distribution function of a $\text{Poisson}(H_t)$ random variable. By the definition of J , the event $\{J(H_t, u_{t+1}) \leq j\}$ is equivalent to $\{u_{t+1} \leq F_{H_t}(j)\}$. Hence, if $u_{t+1} \sim \text{Unif}(0, 1)$, then $\mathbb{P}(J(H_t, u_{t+1}) \leq j) = F_{H_t}(j)$, so $J(H_t, u_{t+1}) \sim \text{Poisson}(H_t)$. Thus, we can write $j_{t+1} = J(H_t, u_{t+1})$.

Economically, the state space contains arbitrarily adverse states: as H increases, the expected number of disasters becomes arbitrarily large and certainty-equivalent consumption growth approaches zero. Thus, recursive utility exists, but we are not able to claim that there is a unique recursive utility $U \in \mathcal{K}(C)$ satisfying $\mathbf{T}U = U$.

However, since all the assumptions of Proposition 15 hold, we are able to claim that the operator $\mathbf{A}$ defined by

$$(\mathbf{A}\varphi)(H) = \left\{ (1 - \beta) + \beta \left[h(H)^s \int_{\mathbb{R}_+} \varphi^s(H') Q(H, dH') \right]^{r/s} \right\}^{1/r},$$

has a unique fixed point, and therefore there is a unique separable recursive utility. Formally, if $\varphi \in \mathcal{K}(\mathbf{1})$ is the unique fixed point of $\mathbf{A}$, then by Proposition 16 the process defined by $U_t = \varphi(H_t)C_t \in \mathcal{K}(C)$ is a recursive utility.

Comparison with the unit-EIS case. It is useful to contrast this uniqueness result with the limit case $r = 0$. Since $h(H) \rightarrow 0$ as $H \rightarrow \infty$, we have that $h \notin \mathcal{K}(\mathbf{1})$ and consequently $\mathcal{M}_t(C_{t+1}) = h_t C_t \notin \mathcal{K}(C)$. Proposition 9 implies that no fixed point belongs to $\mathcal{K}(C)$. Nevertheless, Proposition 11 allows us to identify other admissible recursive utilities. As shown in Appendix S.3, under the calibration of Table 2, equation (15) admits the two solutions

$$Z_t^{(1)} \simeq C_t e^{-83.39H_t}, \quad Z_t^{(2)} \simeq C_t e^{-278.1H_t}.$$

For both solutions, substitution into (15) yields a constant $B_t^{(i)} \equiv \bar{B}_i \in \mathcal{K}(\mathbf{1})$, so that, thanks to (16), we obtain the unique fixed point in each $\mathcal{K}(Z^{(i)})$, given by

$$U_t^{(1)} \simeq C_t e^{0.3686 - 83.39H_t}, \quad U_t^{(2)} \simeq C_t e^{-3.022 - 278.1H_t}.$$

These coincide with the two affine solutions found by Christensen [14, Proposition 2.2]. Observe that $\mathcal{K}(Z^{(1)}) \cap \mathcal{K}(Z^{(2)}) = \emptyset$. Thus, surprisingly, with non-unit EIS there is a unique separable recursive utility in $\mathcal{K}(C)$, whereas for unit EIS two separable recursive utilities exist in distinct cones.

4.7 Long-run risk and multiplicity at unit EIS

Finally, we apply the results of Sections 3.3 and 4.3 to establish multiplicity of separable recursive utilities under the constant-volatility specification of Bansal and Yaron [5]. The proofs are provided in the Supplemental Appendix S.4. Consider the consumption-growth model

$$\log \frac{C_{t+1}}{C_t} = a + X_t + \sigma_t Y_{t+1} \text{ and } X_{t+1} = \rho X_t + \chi \sigma_t \varepsilon_{t+1}^x, \quad (38)$$

where $|\rho| < 1$, $\chi > 0$, and (σ_t) is a time-homogeneous Markov process taking values in $(0, \infty)$. Here, the sequences (ε_t^x) and (Y_t) are each i.i.d., mutually independent,

and independent of (σ_t) as well. We assume that ε_1^x has full support on $\mathbb{R}$, so that X_t has full support on $\mathbb{R}$ for every $t \geq 1$.

Set $r = 0$, $\alpha + \beta = 1$ and take the Epstein–Zin certainty equivalent with $s < 0$. Suppose that $\mathbb{E}[e^{s\sigma Y_1}] < \infty$ holds for every $\sigma > 0$. Under these assumptions the function h is given by

$$h(x, \sigma) = e^{a+x} \left(\mathbb{E}[e^{s\sigma Y_1}] \right)^{1/s}.$$

Since $h(x, \sigma) \rightarrow \infty$ as $x \rightarrow \infty$, it follows that h is unbounded above. Consequently the condition $h \in \mathcal{K}(\mathbf{1})$ of Section 4.3 fails. Whenever $C \in \mathcal{L}_{++}^1(\delta)$, Proposition 9 therefore rules out fixed points in the cone $\mathcal{K}(C)$.

Using Proposition 11, we find two separable recursive utilities for the constant-volatility specification of Bansal and Yaron [5]. Formally, let $\sigma_t := \bar{\sigma} > 0$, $0 < \rho < 1$, and $\varepsilon_t^x, Y_t \sim N(0, 1)$. The two separable recursive utilities are

$$U_t^{(i)} = C_t \exp \left(A_i + \frac{q_i X_t^2 + b_i X_t}{s} \right), \quad i = 0, 1,$$

where $q_0 = 0$ and $q_1 > 0$. For the Bansal–Yaron calibration,

$$\beta = 0.998, \quad \rho = 0.979, \quad a = 0.0015, \quad \bar{\sigma} = 0.0078, \quad \chi = 0.044, \quad s = -9 \quad (39)$$

the separable recursive utilities turn out to be

$$\begin{aligned} U_t^{(0)} &\simeq C_t \exp(0.1121 + 43.47 X_t), \\ U_t^{(1)} &\simeq C_t \exp(-1.219 - 46.53 X_t - (2.051 \times 10^4) X_t^2). \end{aligned}$$

Relative to consumption, the higher utility is log-linear in the state variable, while the lower one is log-quadratic. The existence of a log-linear solution at unit EIS is known (see, e.g., [36]), but our results also identify a second, log-quadratic solution.

Observe that long-run risk models typically assume that the EIS is greater than one (i.e., $r > 0$). In that case, using Proposition 14, the unboundedness of h also rules out a recursive utility in $\mathcal{K}(C)$. We do not expatiate further on this point.

5 Concluding remarks

Uniqueness of recursive utility is a question of central importance, since without it a given set of preference parameters (such as risk aversion, the discount factor, and the elasticity of intertemporal substitution) need not pin down a unique utility, and may therefore imply indeterminate welfare or asset-pricing predictions.

The multiplicity that can arise here resembles a familiar feature of forward-looking equations in economics. For example, in standard monetary models, a weak policy response to inflation can allow multiple self-fulfilling paths for inflation. A similar issue arises here because current utility depends on continuation utility. When the

state variable is persistent, very good or very bad states can last far into the future. Different levels of continuation utility in those distant states can then feed back into utility today, allowing different utility processes to satisfy the same forward-looking equation.

In Markov environments, we have identified a key distinction between unit and non-unit EIS: multiplicity can arise with unit EIS, whereas with non-unit EIS we have established uniqueness within the class of separable recursive utilities. In the long-run-risk model of Bansal and Yaron [5], the state variable ranges over the entire real line, so the bounds required by our unit-EIS uniqueness condition fail. For the constant-volatility specification with unit EIS, we establish multiplicity. Whether such a multiplicity may also arise in this model with non-unit EIS remains an open question.

A Appendix

A.1 Proofs

A.1.1 Proof of Lemma 2

We have seen that $\mathcal{K}(C) \subset L_{++}$. Since

$$\varepsilon \mathcal{M}(C) \leq \mathcal{M}(\varepsilon C) \leq \mathcal{M}(C)$$

for $0 < \varepsilon < 1$, it follows that $\mathcal{M}(\varepsilon C) \in L_+$ and so $\mathbf{T}(\varepsilon C) = W(C, \mathcal{M}(\varepsilon C))$ is well defined.

As $\mathcal{M}$ is subhomogeneous, we can write

$$\frac{\mathbf{T}(\varepsilon C)}{\varepsilon C} = \frac{W(C, \mathcal{M}(\varepsilon C))}{\varepsilon C} \geq \frac{W(C, \varepsilon \mathcal{M}(C))}{\varepsilon C}$$

Moreover, W is positively homogeneous, so

$$\frac{\mathbf{T}(\varepsilon C)}{\varepsilon C} \geq W\left(\frac{1}{\varepsilon}, \frac{\mathcal{M}(C)}{C}\right)$$

Therefore, if we prove that

$$W\left(\frac{1}{\varepsilon}, \frac{\mathcal{M}(C)}{C}\right) \gg 1 \tag{40}$$

we have the desired result. On the other hand,

$$W\left(\frac{1}{\varepsilon}, \frac{\mathcal{M}(C)}{C}\right) = \left[(1 - \beta)\varepsilon^{|r|} + \beta \left(\frac{\mathcal{M}(C)}{C} \right)^r \right]^{1/r}$$

Since $1/r < 0$, (40) is equivalent to

$$(1 - \beta)\varepsilon^{|r|} + \beta \left(\frac{\mathcal{M}(C)}{C} \right)^r \ll 1.$$

After some algebra, we easily obtain the equivalent condition

$$\frac{\mathcal{M}(C)}{C} \gg \left(\frac{1 - (1 - \beta)\varepsilon^{|r|}}{\beta} \right)^{1/r} \quad (41)$$

Observe that

$$\left(\frac{1 - (1 - \beta)\varepsilon^{|r|}}{\beta} \right)^{1/r} \downarrow \beta^{\frac{1}{|r|}} \text{ as } \varepsilon \downarrow 0 \quad (42)$$

From (b) it follows

$$\frac{\mathcal{M}(C)}{C} \geq \beta^{\frac{1}{|r|}} + \eta$$

for some $\eta > 0$, and from (42) we have

$$\left(\frac{1 - (1 - \beta)\varepsilon^{|r|}}{\beta} \right)^{1/r} \leq \beta^{\frac{1}{|r|}} + \eta/2$$

for $\varepsilon > 0$ small enough. At last we get

$$\frac{\mathcal{M}(C)}{C} \geq \beta^{\frac{1}{|r|}} + \eta/2 + \eta/2 \geq \left(\frac{1 - (1 - \beta)\varepsilon^{|r|}}{\beta} \right)^{1/r} + \eta/2$$

which is (41) and so the claim is proved. ■

A.1.2 Proof of Proposition 3

Let $X \in \mathcal{K}(C)$. Then $\mathcal{M}(X) \in L_+$ (see (3)). In view of (4) we have $\mathbf{T}(X) \leq (1 - \beta)^{1/r}C$. On the other hand, $X \geq \varepsilon C$ holds for ε small enough. By Lemma 2 we infer $\mathbf{T}X \geq \mathbf{T}(\varepsilon C) \geq \varepsilon C$ and so we have proved that $\mathbf{T}(X) \in \mathcal{K}(C)$ as well as the relation (8)

Fix now the scalar $\varepsilon > 0$ for which $\mathbf{T}$ maps $[\varepsilon C, (1 - \beta)^{1/r}C]$ into itself, and set $X^0 = (1 - \beta)^{1/r}C$. Then $X^1 = \mathbf{T}X^0 \leq X^0$. As $\mathbf{T}$ is monotone, $\mathbf{T}X^1 = \mathbf{T}^2X^0 \leq \mathbf{T}X^0 \leq X^0$. Recursively, our processes form a decreasing sequence

$$X^0 \geq \dots \geq \mathbf{T}^{n-1}X^0 \geq \mathbf{T}^nX^0 \geq \dots \geq \varepsilon C,$$

which is bounded from below by the element $\varepsilon C \in L_{++}^1(\delta)$. By the Dedekind σ -completeness we have the point-wise convergence

$$\mathbf{T}^nX^0 \downarrow U^* = \inf_n \mathbf{T}^nX^0.$$

As $\mathbf{T}$ is σ -order continuous, $\mathbf{T}U^* = U^*$ and U^* is a fixed point of $\mathbf{T}$ belonging to $L_{++}^1(\delta)$.

Let $U \in \mathcal{K}(C)$ be any fixed point of $\mathbf{T}$. Since $U = \mathbf{T}U$ we have $U \leq (1 - \beta)^{1/r}C$ and, by the monotonicity, $U = \mathbf{T}^n U \leq \mathbf{T}^n((1 - \beta)^{1/r}C)$. Therefore, $U \leq U^*$ as $n \rightarrow \infty$ and this proves that U^* is the greatest fixed point.

Observe that by using the increasing sequence

$$X^0 \leq \dots \leq \mathbf{T}^{n-1}X^0 \leq \mathbf{T}^n X^0 \leq \dots \leq (1 - \beta)^{1/r}C,$$

starting at $X^0 = \varepsilon C$, we deduce, by a similar argument, the existence of a fixed point $U_* = \sup_n \mathbf{T}^n X^0$. The point U_* is the least fixed point in the interval $[\varepsilon C, (1 - \beta)^{1/r}C]$. $\blacksquare$

A.1.3 Proof of Proposition 4

Our aim is to invoke Theorem 1 of [31]. Fix a scalar ε for which (6) holds. Let us prove that no fixed point of $\mathbf{T}$ lies on the lower perimeter $\partial^\circ[\varepsilon C, (1 - \beta)^{1/r}C]$.

Suppose by contradiction that there is a fixed point $\bar{U} = \mathbf{T}\bar{U} \in \partial^\circ[\varepsilon C, (1 - \beta)^{1/r}C]$. Hence $\bar{U} \geq U_*$, where U_* is the least fixed point of $\mathbf{T}$ in the interval $[\varepsilon C, (1 - \beta)^{1/r}C]$ (see the previous proof). Consequently all the elements of the increasing sequence $\mathbf{T}^n(\varepsilon C) \uparrow U_*$ are contained in the lower perimeter. On the other hand, in view of (6) and (10), $\mathbf{T}(\varepsilon C) \notin \partial^\circ[\varepsilon C, (1 - \beta)^{1/r}C]$. This is a contradiction, so Theorem 1 of [31] implies the existence of a unique fixed point in $[\varepsilon C, (1 - \beta)^{1/r}C]$. Since ε can be taken arbitrarily small, we deduce the uniqueness of the fixed point in the cone $\mathcal{K}(C)$. $\blacksquare$

A.1.4 Proof of Proposition 6

Let us only show that the relation $\mathbf{T}((1 - \beta)^{1/r}C)/C \gg (1 - \beta)^{1/r}$ is equivalent to $\mathcal{M}(C)/C \gg 0$. By subhomogeneity,

$$\mathcal{M}((1 - \beta)^{1/r}C) \geq (1 - \beta)^{1/r}\mathcal{M}(C).$$

Since $\mathcal{M}(C)/C \gg 0$, there exists $\eta > 0$ such that $\mathcal{M}(C) \geq \eta C$. Hence

$$\mathcal{M}((1 - \beta)^{1/r}C) \geq (1 - \beta)^{1/r}\eta C.$$

Therefore,

$$\begin{aligned} \frac{\mathbf{T}((1 - \beta)^{1/r}C)}{C} &= \left[(1 - \beta) + \beta \left(\frac{\mathcal{M}((1 - \beta)^{1/r}C)}{C} \right)^r \right]^{1/r} \\ &\geq [(1 - \beta) + \beta(1 - \beta)\eta^r]^{1/r} \\ &> (1 - \beta)^{1/r}, \end{aligned}$$

as desired. The converse is straightforward. The rest of the proof follows the same lines as the proof of Proposition 4. ■

A.1.5 Proof of Proposition 9

Recall that $\mathbf{T}$ is β -subhomogeneous. First, we point out the following equivalence

$$\mathcal{M}(C) \in \mathcal{K}(C^p) \iff \mathbf{T}(C) \in \mathcal{K}(C) \quad (43)$$

Actually, $\mathcal{M}(C) \in \mathcal{K}(C^p)$ means

$$\lambda C^p \leq \mathcal{M}(C) \leq \mu C^p \quad \text{with } \lambda, \mu > 0$$

Straightforward algebra gives

$$\lambda^\beta C^{\alpha+p\beta} \leq \mathbf{T}(C) \leq \mu^\beta C^{\alpha+p\beta}$$

As $\alpha + p\beta = 1$ we deduce that $\mathbf{T}(C) \in \mathcal{K}(C)$. The converse implication is proved similarly.

Suppose now that $\mathcal{M}(C) \in \mathcal{K}(C^p)$. Hence $\mathbf{T}(C) \in \mathcal{K}(C)$ and so

$$mC \leq \mathbf{T}(C) \leq MC$$

for some $0 < m \leq M$. Pick $0 < \varepsilon < 1$ and $\lambda > 1$ such that $\varepsilon^{1-\beta} \leq m$ and $\lambda^{1-\beta} \geq M$. Consequently,

$$\begin{aligned} \mathbf{T}(\varepsilon C) &\geq \varepsilon^\beta \mathbf{T}(C) \geq \varepsilon^\beta mC \geq \varepsilon C \\ \mathbf{T}(\lambda C) &\leq \lambda^\beta \mathbf{T}(C) \leq \lambda^\beta MC \leq \lambda C \end{aligned}$$

Hence $\mathbf{T}$ maps the interval $[\varepsilon C, \lambda C]$ into itself. Since ε and λ can be taken arbitrarily small and large, respectively, $\mathbf{T}$ maps $\mathcal{K}(C)$ into itself. By the usual monotone iteration we deduce the existence of a fixed point in the interval $[\varepsilon C, \lambda C]$.

Let U and V be two fixed points in $\mathcal{K}(C)$. We have thus

$$q^{-1}U \leq V \leq qU$$

for some $q > 1$. Monotonicity and β -subhomogeneity give

$$q^{-\beta} \mathbf{T}U \leq \mathbf{T}(q^{-1}U) \leq \mathbf{T}V \leq \mathbf{T}(qU) \leq q^\beta \mathbf{T}U$$

Using $U = \mathbf{T}U$ and $V = \mathbf{T}V$ yields

$$q^{-\beta}U \leq V \leq q^\beta U$$

Iterating

$$q^{-\beta^n}U \leq V \leq q^{\beta^n}U \quad \text{for every } n$$

Letting $n \rightarrow \infty$ gives $U = V$ and so the fixed point in $\mathcal{K}(C)$ is unique.

Suppose now that $\mathcal{M}(C) \notin \mathcal{K}(C^p)$ and that, by contradiction, there is a fixed point $U \in \mathcal{K}(C)$. Hence $q^{-1}C \leq U \leq qC$ for some $q > 1$. Monotonicity and subhomogeneity give

$$q^{-1}\mathbf{T}C \leq U \leq q\mathbf{T}C$$

Namely $U \in \mathcal{K}(\mathbf{T}C) \iff \mathbf{T}C \in \mathcal{K}(U) = \mathcal{K}(C)$ which contradicts the assumption $\mathcal{M}(C) \notin \mathcal{K}(C^p)$. Consequently, no fixed point lies in $\mathcal{K}(C)$.

A.1.6 Proof of Proposition 11

The analogue of (43) is the equivalence:

$$\mathcal{M}(Z) \in \mathcal{K}(C^{-\alpha/\beta}Z^{1/\beta}) \iff \mathbf{T}Z \in \mathcal{K}(Z)$$

which extends (43). Indeed, if $\mathcal{M}(Z) \in \mathcal{K}(C^{-\alpha/\beta}Z^{1/\beta})$ then $\mathcal{M}_t(Z_{t+1}) = C_t^{-\alpha/\beta}Z_t^{1/\beta}B_t$, where $B \in \mathcal{K}(1)$. Therefore

$$(\mathbf{T}Z)_t = C_t^\alpha [\mathcal{M}_t(Z_{t+1})]^\beta = B_t^\beta Z_t$$

Namely, $\mathbf{T}Z \in \mathcal{K}(Z)$. The converse is similar. At this point the proof goes on exactly as the proof of Proposition 9, by replacing C with Z .

Finally, in the Epstein–Zin case, $\mathcal{M}_t(Z_{t+1}) = (\mathbb{E}_t Z_{t+1}^s)^{1/s}$. Hence condition (14) is equivalent to

$$(\mathbb{E}_t Z_{t+1}^s)^{1/s} = C_t^{-\alpha/\beta} Z_t^{1/\beta} \tilde{B}_t$$

for some $\tilde{B} \in \mathcal{K}(1)$. Raising to the power s and setting $B_t := \tilde{B}_t^s \in \mathcal{K}(1)$ gives (15).

If $B_t \equiv \bar{B} > 0$, then $(\mathbf{T}Z)_t = C_t^\alpha [\mathcal{M}_t(Z_{t+1})]^\beta = C_t^\alpha C_t^{-\alpha} Z_t \bar{B}^{\beta/s} = \bar{B}^{\beta/s} Z_t$. Hence, $\mathbf{T}Z = \bar{B}^{\beta/s} Z$. Setting now $U = \bar{B}^{\frac{\beta}{\alpha s}} Z$, we get

$$\mathbf{T}U = \mathbf{T}\left(\bar{B}^{\frac{\beta}{\alpha s}} Z\right) = \bar{B}^{\frac{\beta^2}{\alpha s}} \mathbf{T}(Z) = \bar{B}^{\frac{\beta^2}{\alpha s}} \bar{B}^{\beta/s} Z = \bar{B}^{\frac{\beta^2}{\alpha s}} \bar{B}^{\beta/s} \bar{B}^{-\frac{\beta}{\alpha s}} U = U$$

as desired. ■

A.1.7 Proof of Proposition 12

Suppose $r < 0$; the other case is similar. Let $\bar{X}$ be any process in $\mathcal{K}(C)$ and $\bar{U}$ be the unique fixed point in $\mathcal{K}(C)$. Consequently there exist a sufficiently small $\varepsilon \in (0, 1)$ and a sufficiently large $\lambda > 1$ such that $\bar{X}, \bar{U} \in [\varepsilon C, \lambda C]$ and with $\mathbf{T}(\varepsilon C) \geq \varepsilon C$, $\mathbf{T}(\lambda C) \leq \lambda C$. Iterating the relation

$$\varepsilon C \leq \bar{X} \leq \lambda C$$

we get $\mathbf{T}^n(\varepsilon C) \leq \mathbf{T}^n(\bar{X}) \leq \mathbf{T}^n(\lambda C)$ and so

$$\mathbf{T}^n(\varepsilon C) - \bar{U} \leq \mathbf{T}^n \bar{X} - \bar{U} \leq \mathbf{T}^n(\lambda C) - \bar{U}$$

Since $\mathbf{T}^n(\varepsilon C) \uparrow \bar{U}$ and $\mathbf{T}^n(\lambda C) \downarrow \bar{U}$ it follows that $\mathbf{T}^n(\varepsilon C) - \bar{U} \leq 0$ and $\mathbf{T}^n(\lambda C) - \bar{U} \geq 0$. This implies

$$|\mathbf{T}^n(\bar{X}) - \bar{U}| \leq Z_n$$

where $Z_n = (\mathbf{T}^n(\lambda C) - \bar{U}) \vee (\bar{U} - \mathbf{T}^n(\varepsilon C))$. Clearly $Z_n \geq 0$ and $Z_{n+1} \downarrow 0$ and this proves the first claim. On the other hand

$$\|\bar{U} - \mathbf{T}^n(\bar{X})\| = \| |\bar{U} - \mathbf{T}^n(\bar{X})| \| \leq \|Z_n\| \downarrow 0$$

by the monotone convergence theorem. ■

A.1.8 Proof of Proposition 13

Let $r \in (0, 1)$ and $U = \mathbf{T}(U)$ be a fixed point. In view of (11) $U \geq (1 - \beta)^{1/r} C$. Let $\bar{\lambda}$ be the greatest scalar such that $U \geq \bar{\lambda} C$ (see Section 2). Therefore,

$$\begin{aligned} U = \mathbf{T}(U) &\geq \mathbf{T}(\bar{\lambda} C) = W(C, \mathcal{M}(\bar{\lambda} C)) = W(C, \bar{\lambda} \mathcal{M}(C)) \\ &= CW \left(1, \bar{\lambda} \frac{\mathcal{M}(C)}{C} \right) = C \left[(1 - \beta) + \beta \bar{\lambda}^r \left[\frac{\mathcal{M}(C)}{C} \right]^r \right]^{1/r} \end{aligned}$$

Inequality (17) implies

$$U \geq C \left[(1 - \beta) + \bar{\lambda}^r \right]^{1/r}$$

Thanks to the property of the scalar $\bar{\lambda}$, we would have

$$\bar{\lambda} \geq \left[(1 - \beta) + \bar{\lambda}^r \right]^{1/r} \iff \bar{\lambda}^r \geq (1 - \beta) + \bar{\lambda}^r$$

which is absurd. Hence, there is no fixed point.

The case $r < 0$ is similar by taking into account that $U \leq (1 - \beta)^{1/r} C$ holds for any fixed point of $\mathbf{T}$. ■

A.1.9 Proof of Proposition 14

Let $r \in (0, 1)$. Suppose, by contradiction, that a fixed point U belongs to $\mathcal{K}(C)$. There is then a constant $q > 1$ such that $q^{-1}C \leq U \leq qC$. Monotonicity and subhomogeneity of the certainty equivalent imply

$$\mathcal{M}_t(U_{t+1}) \geq \mathcal{M}_t(q^{-1}C_{t+1}) \geq q^{-1}\mathcal{M}_t(C_{t+1}).$$

Dividing the recursive-utility equation by C_t and using $r > 0$ therefore gives

$$\begin{aligned} q &\geq \frac{U_t}{C_t} = \left[1 - \beta + \beta \left(\frac{\mathcal{M}_t(U_{t+1})}{C_t} \right)^r \right]^{1/r} \geq \\ &\beta^{1/r} \frac{\mathcal{M}_t(U_{t+1})}{C_t} \geq \beta^{1/r} q^{-1} \frac{\mathcal{M}_t(C_{t+1})}{C_t} \end{aligned}$$

It follows that $\sup \frac{\mathcal{M}(C)}{C} < \infty$. A contradiction.

Analogously, suppose $r < 0$ and $q^{-1}C \leq U \leq qC$ where U is a fixed point of $\mathbf{T}$. Then we can write

$$\mathcal{M}_t(U_{t+1}) \leq \mathcal{M}_t(qC_{t+1}) \leq q\mathcal{M}_t(C_{t+1}).$$

In this case

$$\begin{aligned} q^{-1} &\leq \frac{U_t}{C_t} = \left[1 - \beta + \beta \left(\frac{\mathcal{M}_t(U_{t+1})}{C_t} \right)^r \right]^{1/r} \\ &\leq \beta^{1/r} \frac{\mathcal{M}_t(U_{t+1})}{C_t} \leq \beta^{1/r} q \frac{\mathcal{M}_t(C_{t+1})}{C_t} \end{aligned}$$

and so $\frac{\mathcal{M}(C)}{C} \gg 0$ which contradicts our hypothesis. ■

A.1.10 Proof of Proposition 15

Suppose first that $r \in (0, 1)$ and let $(1 - \beta)^{1/r} \leq \varphi \leq b$. By monotonicity, in view of (25),

$$(1 - \beta)^{1/r} \leq \mathbf{A}\varphi \leq \mathbf{A}b = \Gamma_r(bh(\xi)) \leq \Gamma_r(b\bar{h}) = ((1 - \beta) + \beta b^r \bar{h}^r)^{1/r}.$$

Hence $\mathbf{A}$ maps $[(1 - \beta)^{1/r}, b]$ into itself whenever

$$b \geq \left(\frac{1 - \beta}{1 - \beta \bar{h}^r} \right)^{1/r},$$

which is well defined under (30).

For $r < 0$, an analogous argument, using $\underline{h} = \inf h$, shows that $\mathbf{A}$ maps $[b, (1 - \beta)^{1/r}]$ into itself whenever

$$0 < b \leq \left(\frac{1 - \beta}{1 - \beta \underline{h}^r} \right)^{1/r}.$$

In either case, monotonicity and σ -order continuity imply that the iterates from the endpoints of the invariant interval converge monotonically to its least and greatest fixed points. When $\mathcal{M}$ is order concave, $\mathbf{A}$ is order concave, and uniqueness follows from the same lower-perimeter arguments as in the proofs of Propositions 4 and 6. Finally, for $r \in (0, 1)$ every fixed point belongs to $\mathcal{K}(\mathbf{1})$. ■

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S Supplemental Appendix

S.1 Concavity of the certainty equivalents

We have frequently claimed in the present paper the concavity of certainty equivalents by resorting to a well-established property, known in the finite-dimensional setting: a positively homogeneous and quasi-concave function is concave on the positive cone.

We intend to point out an extension of this basic result to the class of quasi-arithmetic operators

$$\Phi(X) = (\mathbb{E}[\varphi(X)^s \mid \mathcal{F}])^{1/s}$$

Clearly the functional Φ takes values in an ordered vector space and therefore the concepts of concavity as well as quasi-concavity must be adapted. The translation to a Markovian setting is straightforward.

Let $\varphi : B_s \rightarrow L_{++}^0(\Omega, \Sigma, \mathbb{P})$ be a (conditionally) positively homogeneous and concave function defined on a (conditionally) convex set B_s of $L_+^0(\Omega, \Sigma, \mathbb{P})$ such that $\mathbb{E}[\varphi(X)^s \mid \mathcal{F}]$ is finite for every $X \in B_s$ and where $\mathcal{F}$ is a sub- σ -algebra of Σ .

Proposition 17. *The map*

$$\Phi(X) = (\mathbb{E}[\varphi(X)^s \mid \mathcal{F}])^{1/s}$$

from B_s to $L_{++}^0(\Omega, \mathcal{F}, \mathbb{P})$ is positively homogeneous and concave for every $s \in (-\infty, 0) \cup (0, 1)$.

A similar result holds for $s > 1$, provided φ is positively homogeneous and convex. In this case $\Phi(X)$ is convex.

Notice that, by virtue of the property of the expectation, a stronger property of homogeneity will be useful. Namely,

$$\Phi(YX) = Y\Phi(X) \text{ for } X \in B_s \text{ and } Y \in L_{++}^0(\Omega, \mathcal{F}, \mathbb{P}). \quad (44)$$

Earlier, we called this property conditional positive homogeneity in subsection 4.2.

Proof. The map Φ is well defined for every $X \in B_s$, and $\Phi(X)$ is a finite and strictly positive $\mathcal{F}$ -measurable random variable for every $X \in B_s$.

Let us first assume $s \in (0, 1)$.

It is trivial to check that the map $X \mapsto \varphi(X)^s$ from B_s into L_{++}^0 turns out to be concave for $s \in (0, 1)$. Consider first the map

$$\rho(X) = \mathbb{E}[\varphi(X)^s \mid \mathcal{F}]$$

defined on B_s . From the properties of the conditional expectation (see for instance [34, Sect. IV.3]) it follows that the concavity is preserved. Hence,

$$\rho(\beta X_1 + \bar{\beta} X_2) \geq \beta \rho(X_1) + \bar{\beta} \rho(X_2)$$

for all $X_1, X_2 \in B_s$ and $\bar{\beta} = 1 - \beta$.

We show that ρ is quasi-concave in the following sense. Given any $Y \in L_{++}^0(\Omega, \mathcal{F}, \mathbb{P})$, define the upper-level set

$$U_\rho(Y) = \{X \in B_s : \rho(X) \geq Y\}$$

It is easy to check that $U_\rho(Y)$ is convex whenever it is not empty.

Since $\Phi(X) = [\rho(X)]^{1/s}$, we have $U_\Phi(Y) = U_\rho(Y^s)$ because $X \mapsto X^s$ is strictly increasing. Hence $\Phi(X) = (\mathbb{E}[\varphi(X)^s | \mathcal{F}])^{1/s}$ is quasi-concave as well.

Now let $Y_i = \Phi(X_i)$ for $X_i \in B_s$ and $i = 1, 2$. In view of (44), we have $\Phi(X_i/Y_i) = \Phi(X_i)/Y_i = 1$. Consequently, $X_i/Y_i \in U_\Phi(1)$. But $U_\Phi(1)$ is convex. Hence

$$\Phi\left(\frac{Y_1}{Y_1 + Y_2} \cdot \frac{X_1}{Y_1} + \frac{Y_2}{Y_1 + Y_2} \cdot \frac{X_2}{Y_2}\right) = \Phi\left(\frac{X_1 + X_2}{Y_1 + Y_2}\right) \geq 1$$

and hence $\Phi(X_1 + X_2) \geq \Phi(X_1) + \Phi(X_2)$. This is superadditivity; combined with positive homogeneity, it implies concavity.

When $s < 0$, the map $X \mapsto \varphi(X)^s$ is convex, so ρ is convex. Because $t \mapsto t^{1/s}$ is decreasing, Φ is again quasi-concave and the proof proceeds in the same way. $\blacksquare$

S.2 Calculations for the hidden-regime model

Here, we formally recast the two-state hidden-regime model from Ju and Miao [23] into our Markov economy notation and then verify the calculations reported in Section 4.5 (see also Section 3 in [23]).

Primitive hidden-regime economy. Let $S_t \in \{1, 2\}$ be the unobserved regime, with transition probabilities

$$\mathbb{P}(S_{t+1} = 1 | S_t = 1) = \lambda_{11}, \quad \mathbb{P}(S_{t+1} = 1 | S_t = 2) = \lambda_{21}, \quad 0 < \lambda_{21} < \lambda_{11} < 1.$$

At date t , the agent has observed the growth history $\mathcal{Y}_t = \sigma(g_1, \dots, g_t)$, but not the underlying regime. The agent's belief at time t that the economy will be in the high-growth regime in the next period is

$$\mu_t := \mathbb{P}(S_{t+1} = 1 | \mathcal{Y}_t).$$

Conditional on $S_{t+1} = i$, the density of g_{t+1} is f_i . The conditional density of g_{t+1} given the information available at date t is therefore obtained from the law of total probability:

$$\begin{aligned} f_{g_{t+1}|\mathcal{Y}_t}(g) &= \sum_{i=1}^2 \mathbb{P}(S_{t+1} = i | \mathcal{Y}_t) f_{g_{t+1}|S_{t+1}=i, \mathcal{Y}_t}(g) \\ &= \mu_t f_1(g) + (1 - \mu_t) f_2(g) := f_{\mu_t}(g). \end{aligned}$$

After observing $g_{t+1} = g$, the agent updates the belief $p(\mu_t, g)$ that the regime realized at date $t + 1$ was the high-growth regime. Since $\mathcal{Y}_{t+1} = \sigma(\mathcal{Y}_t, g_{t+1})$, Bayes' rule gives

$$\begin{aligned} p(\mu_t, g) &:= \mathbb{P}(S_{t+1} = 1 \mid \mathcal{Y}_t, g_{t+1} = g) \\ &= \frac{\mathbb{P}(S_{t+1} = 1 \mid \mathcal{Y}_t) f_{g_{t+1} \mid S_{t+1}=1, \mathcal{Y}_t}(g)}{f_{g_{t+1} \mid \mathcal{Y}_t}(g)} \\ &= \frac{\mu_t f_1(g)}{\mu_t f_1(g) + (1 - \mu_t) f_2(g)} = \frac{\mu_t f_1(g)}{f_{\mu_t}(g)}. \end{aligned} \quad (45)$$

For μ_{t+1} we obtain

$$\begin{aligned} \mu_{t+1} &= \sum_{i=1}^2 \mathbb{P}(S_{t+2} = 1 \mid S_{t+1} = i, \mathcal{Y}_{t+1}) \mathbb{P}(S_{t+1} = i \mid \mathcal{Y}_{t+1}) \\ &= \lambda_{11} p(\mu_t, g_{t+1}) + \lambda_{21} [1 - p(\mu_t, g_{t+1})] \\ &= \lambda_{21} + (\lambda_{11} - \lambda_{21}) p(\mu_t, g_{t+1}) := B(\mu_t, g_{t+1}). \end{aligned} \quad (46)$$

Now, because $0 < \mu < 1$ and the Gaussian densities $f_1(g)$ and $f_2(g)$ are strictly positive for every finite g , Bayes' formula implies $p(\mu, g) \in (0, 1)$, and since $0 < \lambda_{21} < \lambda_{11} < 1$, it follows that

$$\lambda_{21} < B(\mu, g) < \lambda_{11}.$$

Hence, we can conclude that $\mu_t \in (\lambda_{21}, \lambda_{11})$ for every $t \geq 1$. Therefore, one can take its closure $\mathcal{Z} = K := [\lambda_{21}, \lambda_{11}]$ as the state space.

Function κ . Note that equation (46) is equivalent to

$$\frac{p(\mu_t, g_{t+1})}{1 - p(\mu_t, g_{t+1})} = \frac{\mu_{t+1} - \lambda_{21}}{\lambda_{11} - \mu_{t+1}},$$

which using (45) yields

$$\frac{f_1(g_{t+1})}{f_2(g_{t+1})} = \frac{1 - \mu_t}{\mu_t} \frac{\mu_{t+1} - \lambda_{21}}{\lambda_{11} - \mu_{t+1}}.$$

Using the fact that f_1 and f_2 are normal densities with a common variance, we obtain

$$\log \frac{f_1(g_{t+1})}{f_2(g_{t+1})} = \frac{\bar{g}_1 - \bar{g}_2}{\sigma^2} \left(g_{t+1} - \frac{\bar{g}_1 + \bar{g}_2}{2} \right).$$

Therefore, combining these two identities gives

$$\log \left(\frac{1 - \mu_t}{\mu_t} \right) + \log \left(\frac{\mu_{t+1} - \lambda_{21}}{\lambda_{11} - \mu_{t+1}} \right) = \frac{\bar{g}_1 - \bar{g}_2}{\sigma^2} \left(g_{t+1} - \frac{\bar{g}_1 + \bar{g}_2}{2} \right),$$

which, when solved for $g_{t+1} = \kappa(\mu_t, \mu_{t+1})$, yields

$$\kappa(\mu_t, \mu_{t+1}) = \frac{\bar{g}_1 + \bar{g}_2}{2} + \frac{\sigma^2}{\bar{g}_1 - \bar{g}_2} \left[\log \frac{\mu_{t+1} - \lambda_{21}}{\lambda_{11} - \mu_{t+1}} - \log \frac{\mu_t}{1 - \mu_t} \right].$$

Thus, conditional on the current belief, the next belief and realized growth contain the same information. This is the function used in (34).

Transition kernel Q . We want to find $\mathbb{P}(\mu_{t+1} \in A \mid \mu_0, \dots, \mu_t)$. Using the fact that $\mu_{t+1} = B(\mu_t, g_{t+1})$ and the law of iterated expectations, we can write

$$\begin{aligned} \mathbb{P}(\mu_{t+1} \in A \mid \mu_0, \dots, \mu_t) &= \mathbb{E} [\mathbb{P}(\mu_{t+1} \in A \mid \mathcal{Y}_t) \mid \mu_0, \dots, \mu_t] \\ &= \mathbb{E} \left[\int_{\mathbb{R}} \mathbf{1}_A(B(\mu_t, g)) f_{\mu_t}(g) dg \mid \mu_0, \dots, \mu_t \right] \\ &= \int_{\mathbb{R}} \mathbf{1}_A(B(\mu_t, g)) f_{\mu_t}(g) dg = \mathbb{P}(\mu_{t+1} \in A \mid \mu_t). \end{aligned}$$

Therefore, $(\mu_t)_{t \geq 0}$ is a time-homogeneous Markov process with transition kernel Q given by

$$Q(\mu_t, A) := \mathbb{P}(\mu_{t+1} \in A \mid \mu_t) = \int_{\mathbb{R}} \mathbf{1}_A(B(\mu_t, g)) f_{\mu_t}(g) dg.$$

For each fixed μ , the map $g \mapsto B(\mu, g)$ is a continuously differentiable and strictly increasing bijection from $\mathbb{R}$ onto $(\lambda_{21}, \lambda_{11})$. Hence, it is immediate to verify that $Q(\mu, \cdot)$ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative is

$$\frac{dQ(\mu, \cdot)}{d\mu'}(\mu') = f_{\mu}(\kappa(\mu, \mu')) \frac{\partial \kappa(\mu, \mu')}{\partial \mu'} = f_{\mu}(\kappa(\mu, \mu')) J(\mu, \mu'),$$

for every $\mu' \in (\lambda_{21}, \lambda_{11})$. Hence

$$Q(\mu, d\mu') = f_{\mu}(\kappa(\mu, \mu')) J(\mu, \mu') d\mu',$$

or, equivalently, for every Borel set $A \subseteq K$,

$$Q(\mu, A) = \int_A f_{\mu}(\kappa(\mu, \mu')) J(\mu, \mu') d\mu',$$

as desired.

Integrability of consumption. Since $C_t = C_0 \exp \left(\sum_{j=1}^t g_j \right)$, C_t is $\mathcal{Y}_t$ -measurable. In particular, we have that

$$\mathbb{E}[C_{t+1} \mid \mathcal{Y}_t] = C_t \mathbb{E}[e^{g_{t+1}} \mid \mathcal{Y}_t] = C_t \left\{ \mu_t \mathbb{E}[e^{\bar{g}_1 + \sigma \varepsilon_{t+1}}] + (1 - \mu_t) \mathbb{E}[e^{\bar{g}_2 + \sigma \varepsilon_{t+1}}] \right\}.$$

As $\varepsilon_{t+1} \sim N(0, 1)$, we get

$$\mathbb{E}[e^{\bar{g}_i + \sigma \varepsilon_{t+1}}] = \exp\left(\bar{g}_i + \frac{1}{2}\sigma^2\right), \quad i = 1, 2.$$

Thus

$$\mathbb{E}[C_{t+1} \mid \mathcal{Y}_t] = C_t \exp\left(\frac{1}{2}\sigma^2\right) [\mu_t e^{\bar{g}_1} + (1 - \mu_t) e^{\bar{g}_2}]. \quad (47)$$

Since $\bar{g}_1 > \bar{g}_2$ and $\mu_t \leq \lambda_{11}$, equation (47) gives

$$\mathbb{E}[C_{t+1} \mid \mathcal{Y}_t] \leq C_t \exp\left(\frac{1}{2}\sigma^2\right) [\lambda_{11} e^{\bar{g}_1} + (1 - \lambda_{11}) e^{\bar{g}_2}].$$

Taking expectations and using the law of iterated expectations,

$$\mathbb{E}[C_{t+1}] = \mathbb{E}[\mathbb{E}[C_{t+1} \mid \mathcal{Y}_t]] \leq \exp\left(\frac{1}{2}\sigma^2\right) [\lambda_{11} e^{\bar{g}_1} + (1 - \lambda_{11}) e^{\bar{g}_2}] \mathbb{E}[C_t].$$

Iterating this inequality yields

$$\mathbb{E}[C_t] \leq C_0 \left(\exp\left(\frac{1}{2}\sigma^2\right) [\lambda_{11} e^{\bar{g}_1} + (1 - \lambda_{11}) e^{\bar{g}_2}] \right)^t.$$

Consequently,

$$\sum_{t \geq 0} \delta^t \mathbb{E}[C_t] < \infty \quad \text{whenever} \quad \delta \exp\left(\frac{1}{2}\sigma^2\right) [\lambda_{11} e^{\bar{g}_1} + (1 - \lambda_{11}) e^{\bar{g}_2}] < 1,$$

as desired.

S.2.1 Smooth ambiguity model

Growth factor. For $i = 1, 2$, the regime-conditional certainty equivalent of consumption is

$$\mathcal{M}_t^i(C_{t+1}) = C_t \left[\int_K e^{s\kappa(\mu_t, \mu')} Q_i(\mu_t, d\mu') \right]^{1/s}.$$

By the density representation of Q_i and the change of variables $g = \kappa(\mu, \mu')$,

$$\int_K e^{s\kappa(\mu, \mu')} Q_i(\mu, d\mu') = \int_{\mathbb{R}} e^{sg} f_i(g) dg = \exp\left(s\bar{g}_i + \frac{1}{2}s^2\sigma^2\right).$$

Hence, with

$$a_i := \bar{g}_i + \frac{1}{2}s\sigma^2,$$

we have $\mathcal{M}_t^i(C_{t+1}) = C_t e^{a_i}$. It follows that the smooth-ambiguity certainty equivalent satisfies

$$\frac{\mathcal{M}_t(C_{t+1})}{C_t} = h(\mu_t),$$

with

$$\begin{aligned} h(\mu) &= \{\mu e^{\eta a_1} + (1 - \mu) e^{\eta a_2}\}^{1/\eta} \\ &= \left\{ \mu \exp \left[\eta \left(\bar{g}_1 + \frac{1}{2} s \sigma^2 \right) \right] + (1 - \mu) \exp \left[\eta \left(\bar{g}_2 + \frac{1}{2} s \sigma^2 \right) \right] \right\}^{1/\eta}. \end{aligned}$$

Moreover,

$$h'(\mu) = \frac{e^{\eta a_1} - e^{\eta a_2}}{\eta} [\mu e^{\eta a_1} + (1 - \mu) e^{\eta a_2}]^{1/\eta - 1} > 0,$$

as $a_1 > a_2$. Thus h is increasing for every $\eta \neq 0$, and its minimum and maximum on K are attained at λ_{21} and λ_{11} , respectively.

S.3 Rare-disaster economy

This appendix checks the formulas of Section 4.6. Recall that for every $n \geq 0$,

$$\int_0^\infty H' \frac{(H')^{d_H + n - 1} e^{-H'/c}}{c^{d_H + n} \Gamma(d_H + n)} dH' = c(d_H + n),$$

where for every $a > 0$

$$\Gamma(a) := \int_0^\infty x^{a-1} e^{-x} dx.$$

Disaster-intensity process. If $q(H, H')$ denotes the conditional density defined in (36), the associated transition kernel is

$$Q(H, A) = \int_A q(H, H') dH', \quad A \in \mathcal{B}(\mathbb{R}_+).$$

Consequently,

$$\begin{aligned} \mathbb{E}[H_{t+1} \mid H_t = H] &= \int_{\mathbb{R}_+} H' Q(H, dH') \\ &= \int_0^\infty H' q(H, H') dH' \\ &= \int_0^\infty H' e^{-\phi H/c} \sum_{n=0}^\infty \frac{(\phi H/c)^n}{n!} \frac{(H')^{d_H + n - 1} e^{-H'/c}}{c^{d_H + n} \Gamma(d_H + n)} dH' \\ &= e^{-\phi H/c} \sum_{n=0}^\infty \frac{(\phi H/c)^n}{n!} \int_0^\infty H' \frac{(H')^{d_H + n - 1} e^{-H'/c}}{c^{d_H + n} \Gamma(d_H + n)} dH' \\ &= e^{-\phi H/c} \sum_{n=0}^\infty \frac{(\phi H/c)^n}{n!} c(d_H + n). \end{aligned}$$

Given this, we obtain

$$\begin{aligned} \mathbb{E}[H_{t+1} \mid H_t = H] &= c e^{-\phi H/c} \sum_{n=0}^\infty \frac{(\phi H/c)^n}{n!} (d_H + n) \\ &= c d_H + c e^{-\phi H/c} \sum_{n=0}^\infty n \frac{(\phi H/c)^n}{n!} = c d_H + \phi H, \end{aligned}$$

where we are using $e^{-\lambda} \sum_{n=0}^{\infty} n\lambda^n/n! = \lambda$. Therefore,

$$\mathbb{E}[H_{t+1} \mid H_t] = cd_H + \phi H_t,$$

as desired. Moreover, one can also show that (see [17, Appendix 1, Lemma 1])

$$\mathbb{E}\left[e^{bH_{t+1}} \mid H_t = H\right] = (1 - cb)^{-d_H} \exp\left(\frac{\phi b}{1 - cb} H\right), \quad cb < 1. \quad (48)$$

Integrability of consumption. Using the law of iterated expectations and conditioning first on j_{t+1} , we obtain

$$\begin{aligned} \mathbb{E}[e^{g_{t+1}} \mid H_t = H] &= e^{\nu_g} \mathbb{E}\left[e^{\nu_j j_{t+1}} \mathbb{E}\left[e^{\sigma_j \sqrt{j_{t+1}} \eta_{t+1} + \sigma \varepsilon_{t+1}} \mid j_{t+1}, H_t = H\right] \mid H_t = H\right] \\ &= \exp\left(\nu_g + \frac{1}{2}\sigma^2\right) \mathbb{E}\left[\exp\left\{\left(\nu_j + \frac{1}{2}\sigma_j^2\right) j_{t+1}\right\} \mid H_t = H\right]. \end{aligned}$$

The second equality uses the independence and standard normality of η_{t+1} and ε_{t+1} , together with $\mathbb{E}[e^{aX}] = e^{a^2/2}$ for $X \sim N(0, 1)$.

Since $j_{t+1} \mid H_t = H \sim \text{Poisson}(H)$,

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\left(\nu_j + \frac{1}{2}\sigma_j^2\right) j_{t+1}\right\} \mid H_t = H\right] \\ &= e^{-H} \sum_{k=0}^{\infty} \frac{[H \exp(\nu_j + \frac{1}{2}\sigma_j^2)]^k}{k!} \\ &= \exp\left\{H \left[\exp\left(\nu_j + \frac{1}{2}\sigma_j^2\right) - 1\right]\right\}. \end{aligned}$$

Therefore,

$$\mathbb{E}[e^{g_{t+1}} \mid H_t = H] = \exp\left(\nu_g + \frac{1}{2}\sigma^2\right) \exp\left\{H \left[\exp\left(\nu_j + \frac{1}{2}\sigma_j^2\right) - 1\right]\right\}. \quad (49)$$

If

$$\nu_j + \frac{1}{2}\sigma_j^2 \leq 0,$$

then

$$\exp\left(\nu_j + \frac{1}{2}\sigma_j^2\right) - 1 \leq 0,$$

and therefore

$$\mathbb{E}[e^{g_{t+1}} \mid H_t] \leq \exp\left(\nu_g + \frac{1}{2}\sigma^2\right). \quad (50)$$

Since $C_{t+1} = C_t e^{g_{t+1}}$, the law of iterated expectations and (50) imply

$$\begin{aligned}\mathbb{E}[C_{t+1} \mid H_0] &= \mathbb{E}\left[C_t \mathbb{E}[e^{g_{t+1}} \mid H_t] \mid H_0\right] \\ &\leq \exp\left(\nu_g + \frac{1}{2}\sigma^2\right) \mathbb{E}[C_t \mid H_0].\end{aligned}$$

Iterating this inequality yields

$$\mathbb{E}[C_t \mid H_0] \leq C_0 \exp\left\{t\left(\nu_g + \frac{1}{2}\sigma^2\right)\right\}.$$

Consequently,

$$\begin{aligned}\sum_{t=0}^{\infty} \delta^t \mathbb{E}[C_t \mid H_0] &\leq C_0 \sum_{t=0}^{\infty} \left[\delta \exp\left(\nu_g + \frac{1}{2}\sigma^2\right)\right]^t \\ &< \infty\end{aligned}$$

whenever

$$\delta \exp\left(\nu_g + \frac{1}{2}\sigma^2\right) < 1.$$

A sufficient condition for $C \in \mathcal{L}_{++}^1(\delta)$ is

$$\nu_j + \frac{1}{2}\sigma_j^2 \leq 0 \quad \text{and} \quad 0 < \delta < \min\left\{1, \exp\left(-\nu_g - \frac{1}{2}\sigma^2\right)\right\},$$

as desired.

Function h . Using (37) and the same calculation as in (49) we have

$$\begin{aligned}h(H)^s &= \mathbb{E}[e^{sg_{t+1}} \mid H_t = H] = \exp\left(s\nu_g + \frac{1}{2}s^2\sigma^2\right) \\ &\quad \times \exp\left\{H \left[\exp\left(s\nu_j + \frac{1}{2}s^2\sigma_j^2\right) - 1\right]\right\}.\end{aligned}$$

Namely,

$$h(H) = \exp\left\{\nu_g + \frac{1}{2}s\sigma^2 + \frac{H}{s} \left[\exp\left(s\nu_j + \frac{1}{2}s^2\sigma_j^2\right) - 1\right]\right\}.$$

Defining

$$B_s := \exp\left(\nu_g + \frac{1}{2}s\sigma^2\right), \quad A_s := \frac{\exp\left(s\nu_j + \frac{1}{2}s^2\sigma_j^2\right) - 1}{s},$$

we get

$$h(H) = B_s e^{A_s H},$$

as desired.

Fixed points. In this case the operator $\mathbf{A}$ of Section 4.2 leads to

$$\begin{aligned} (\mathbf{A}\varphi)(H) &= \left\{ (1 - \beta) + \beta \left[\int_{\mathbb{R}_+} \int_{\mathcal{Y}} e^{s\kappa(H,y)} \varphi(H')^s \nu(dy) Q(H, dH') \right]^{r/s} \right\}^{1/r} \\ &= \left\{ (1 - \beta) + \beta \left[\left(\int_{\mathcal{Y}} e^{s\kappa(H,y)} \nu(dy) \right) \left(\int_{\mathbb{R}_+} \varphi(H')^s Q(H, dH') \right) \right]^{r/s} \right\}^{1/r} \\ &= \left\{ (1 - \beta) + \beta \left[h(H)^s \int_{\mathbb{R}_+} \varphi(H')^s Q(H, dH') \right]^{r/s} \right\}^{1/r}. \end{aligned}$$

Hence φ is a fixed point of $\mathbf{A}$ if

$$\varphi(H) = \left\{ (1 - \beta) + \beta \left[h(H)^s \int_{\mathbb{R}_+} \varphi(H')^s Q(H, dH') \right]^{r/s} \right\}^{1/r}.$$

By Proposition 16, every such positive fixed point φ generates a separable recursive utility

$$U_t = \varphi(H_t) C_t.$$

As $\lim_{H \rightarrow \infty} h(H) = 0$, it follows that any fixed point φ of $\mathbf{A}$ must satisfy $\lim_{H \rightarrow \infty} \varphi(H) = (1 - \beta)^{1/r}$, which is consistent with the fact that $\varphi \in \mathcal{K}(\mathbf{1})$.

Unit-EIS solutions. Consider the ansatz

$$Z_t = C_t \exp\left(\frac{b}{s} H_t\right),$$

where b is an unknown parameter to be determined. Since, conditional on H_t , the shocks determining consumption growth are independent of H_{t+1} ,

$$\mathbb{E}_t[Z_{t+1}^s] = C_t^s \mathbb{E}_t[e^{sg_{t+1}}] \mathbb{E}_t[e^{bH_{t+1}}].$$

Using $h(H) = (\mathbb{E}[e^{sg_{t+1}} \mid H_t = H])^{1/s} = B_s e^{A_s H}$ and (48), we obtain

$$\mathbb{E}_t[Z_{t+1}^s] = C_t^s B_s^s (1 - cb)^{-d_H} \exp\left\{\left(sA_s + \frac{\phi b}{1 - cb}\right) H_t\right\},$$

for $cb < 1$. The right-hand side of (15) is

$$C_t^{-\alpha s/\beta} Z_t^{s/\beta} B_t = C_t^s \exp\left(\frac{b}{\beta} H_t\right) B_t.$$

It follows that

$$B_t = B_s^s (1 - cb)^{-d_H} \exp\left\{\left(sA_s + \frac{\phi b}{1 - cb} - \frac{b}{\beta}\right) H_t\right\}.$$

Since H_t has unbounded support on $\mathbb{R}_+$, we have $B \in \mathcal{K}(\mathbf{1})$ if and only if the coefficient of H_t in the exponent is zero, that is,

$$sA_s + \frac{\phi b}{1 - cb} = \frac{b}{\beta},$$

or, equivalently,

$$cb^2 + (\beta\phi - 1 - \beta sA_s c)b + \beta sA_s = 0.$$

Under the calibration of Table 2, the two roots are

$$b_1 \simeq 231.8, \quad b_2 \simeq 773.2,$$

and therefore

$$Z_t^{(1)} \simeq C_t e^{-83.39H_t}, \quad Z_t^{(2)} \simeq C_t e^{-278.1H_t}.$$

For each root, the corresponding process in (15) is constant,

$$B_t^{(i)} \equiv \bar{B}_i = B_s^s (1 - cb_i)^{-d_H}.$$

Hence, in view of (16), the unique recursive utilities in the two corresponding cones are

$$U^{(i)} = \bar{B}_i^{\beta/[s(1-\beta)]} Z^{(i)}, \quad i = 1, 2,$$

which numerically yield the two utilities

$$U_t^{(1)} \simeq C_t e^{0.3686 - 83.39H_t}, \quad U_t^{(2)} \simeq C_t e^{-3.022 - 278.1H_t}.$$

S.4 Calculations for the long-run risk model

Here we prove the claims in Section 4.7. In the constant-volatility specification we write $\sigma_t \equiv \bar{\sigma} > 0$, consistently with the calibration in (39). Throughout, $C_0 > 0$ and $X_0 \in \mathbb{R}$ are assumed to be deterministic.

Failure of uniqueness. Let us first describe the general model (38). Let Q_{vol} be the transition kernel of the Markov process (σ_t) and η the distribution of ε_1^x . The Markov state (X_t, σ_t) has transition kernel

$$Q((x, \sigma), A \times B) = Q_{\text{vol}}(\sigma, B) \int_{\mathbb{R}} \mathbf{1}_A(\rho x + \chi \sigma e) \eta(de),$$

for Borel sets $A \subseteq \mathbb{R}$ and $B \subseteq (0, \infty)$. The growth function κ in (22) is

$$\kappa((x, \sigma), (x', \sigma'), y') = \mu_c + x + \sigma y'.$$

Under our assumptions, we have that

$$\mathbb{E}_t[C_{t+1}^s] = C_t^s e^{s(a+X_t)} \mathbb{E}_t[e^{s\sigma_t Y_{t+1}}],$$

which implies

$$\frac{\mathcal{M}_t(C_{t+1})}{C_t} = e^{a+X_t} \left(\mathbb{E}_t[e^{s\sigma_t Y_{t+1}}] \right)^{1/s} = h(X_t, \sigma_t).$$

Therefore,

$$h(x, \sigma) = e^{a+x} \left(\mathbb{E}[e^{s\sigma Y_1}] \right)^{1/s}.$$

Observe that $h(x, \sigma) \rightarrow \infty$ as $x \rightarrow \infty$.

Constant-volatility model. For the rest of the construction, let $0 < \rho < 1$ and $\sigma_t := \bar{\sigma} > 0$, and take both innovation sequences to be standard normal. Write $\sigma_x = \chi \bar{\sigma} > 0$. The dynamic model becomes

$$\log \frac{C_{t+1}}{C_t} = a + X_t + \bar{\sigma} Y_{t+1}, \quad X_{t+1} = \rho X_t + \sigma_x \varepsilon_{t+1}^x.$$

Given the ansatz

$$Z_t = C_t \exp\left(\frac{qX_t^2 + bX_t}{s}\right)$$

the factor multiplying B_t on the right-hand side of (15) is given by

$$\begin{aligned} C_t^{-(1-\beta)s/\beta} Z_t^{s/\beta} &= C_t^{-(1-\beta)s/\beta + s/\beta} \exp\left(\frac{qX_t^2 + bX_t}{\beta}\right) \\ &= C_t^s \exp\left(\frac{qX_t^2 + bX_t}{\beta}\right). \end{aligned}$$

The left-hand side of equation (15) is

$$\mathbb{E}_t[Z_{t+1}^s] = C_t^s e^{s(a+X_t)} \mathbb{E}[e^{s\bar{\sigma} Y_1}] \mathbb{E}_t[e^{qX_{t+1}^2 + bX_{t+1}}].$$

Observe that we have

$$\begin{aligned} \mathbb{E}[e^{s\bar{\sigma} Y_1}] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{s\bar{\sigma} y - y^2/2} dy \\ &= e^{s^2 \bar{\sigma}^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(y - s\bar{\sigma})^2/2} dy = e^{s^2 \bar{\sigma}^2/2}. \end{aligned}$$

For the state innovation, put $d = 1 - 2q\sigma_x^2$ and assume $d > 0$. Conditional on $X_t = x$, the next state has the form $X_{t+1} = \rho x + \sigma_x \varepsilon_{t+1}^x$, where ε_{t+1}^x is standard normal. Hence

$$\mathbb{E}_t[e^{qX_{t+1}^2 + bX_{t+1}}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(q(\rho x + \sigma_x y)^2 + b(\rho x + \sigma_x y) - \frac{y^2}{2}\right) dy.$$

Expanding the exponent gives

$$\begin{aligned} &q(\rho x + \sigma_x y)^2 + b(\rho x + \sigma_x y) - \frac{y^2}{2} \\ &= q\rho^2 x^2 + b\rho x + \left(q\sigma_x^2 - \frac{1}{2}\right) y^2 + \sigma_x(2q\rho x + b)y. \end{aligned}$$

Since $d = 1 - 2q\sigma_x^2 > 0$, the coefficient of y^2 is $-d/2$. Taking the terms independent of y outside the integral therefore yields

$$\mathbb{E}_t[e^{qX_{t+1}^2 + bX_{t+1}}] = \frac{e^{q\rho^2 x^2 + b\rho x}}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{d}{2}y^2 + \sigma_x(2q\rho x + b)y\right) dy.$$

To evaluate this integral, set $m = \sigma_x(2q\rho x + b)/d$ and complete the square:

$$-\frac{d}{2}y^2 + \sigma_x(2q\rho x + b)y = -\frac{d}{2}(y - m)^2 + \frac{d}{2}m^2.$$

The change of variables $u = \sqrt{d}(y - m)$ gives

$$\int_{\mathbb{R}} e^{-d(y-m)^2/2} dy = \frac{1}{\sqrt{d}} \int_{\mathbb{R}} e^{-u^2/2} du = \sqrt{\frac{2\pi}{d}}.$$

Consequently,

$$\mathbb{E}_t[e^{qX_{t+1}^2 + bX_{t+1}}] = d^{-1/2} \exp\left(q\rho^2 x^2 + b\rho x + \frac{\sigma_x^2(2q\rho x + b)^2}{2d}\right).$$

Finally, putting the exponent over the common denominator d and using $d + 2q\sigma_x^2 = 1$, its numerator simplifies to

$$\begin{aligned} & d(q\rho^2 x^2 + b\rho x) + \frac{\sigma_x^2}{2}(2q\rho x + b)^2 \\ &= (d + 2q\sigma_x^2)(q\rho^2 x^2 + b\rho x) + \frac{\sigma_x^2 b^2}{2} \\ &= q\rho^2 x^2 + b\rho x + \frac{\sigma_x^2 b^2}{2}. \end{aligned}$$

We therefore obtain

$$\mathbb{E}_t[e^{qX_{t+1}^2 + bX_{t+1}}] = d^{-1/2} \exp\left(\frac{q\rho^2 X_t^2 + b\rho X_t + \frac{1}{2}\sigma_x^2 b^2}{d}\right),$$

which implies

$$\begin{aligned} \mathbb{E}_t[Z_{t+1}^s] &= C_t^s \exp\left(sa + \frac{s^2 \bar{\sigma}^2}{2}\right) d^{-1/2} \\ &\quad \times \exp\left(sX_t + \frac{q\rho^2 X_t^2 + b\rho X_t + \frac{1}{2}\sigma_x^2 b^2}{d}\right). \end{aligned}$$

Dividing this expression by $C_t^s \exp((qX_t^2 + bX_t)/\beta)$, we find that the process B in (15) satisfies:

$$\begin{aligned} \log B_t &= sa + \frac{s^2 \bar{\sigma}^2}{2} - \frac{1}{2} \log d + \frac{\sigma_x^2 b^2}{2d} \\ &\quad + q\left(\frac{\rho^2}{d} - \frac{1}{\beta}\right) X_t^2 + \left(s + \frac{b\rho}{d} - \frac{b}{\beta}\right) X_t. \end{aligned}$$

Hence $B \in \mathcal{K}(\mathbf{1})$ is equivalent to

$$q \left(\frac{\rho^2}{d} - \frac{1}{\beta} \right) = 0, \quad s + \frac{b\rho}{d} - \frac{b}{\beta} = 0.$$

If $q = 0$, then $d = 1$ and the second equation gives $b = \beta s / (1 - \beta\rho)$. If $q \neq 0$, the first equation gives $d = \beta\rho^2$. In view of the definition of d we get $q = (1 - \beta\rho^2)/(2\sigma_x^2)$, while the second equation becomes

$$s + \frac{b}{\beta\rho} - \frac{b}{\beta} = 0 \iff \beta s\rho + b(1 - \rho) = 0.$$

The two choices are therefore

$$(q_0, b_0) = \left(0, \frac{\beta s}{1 - \beta\rho} \right), \quad (q_1, b_1) = \left(\frac{1 - \beta\rho^2}{2\sigma_x^2}, \frac{\beta s\rho}{\rho - 1} \right).$$

They satisfy $q_1 > 0$, $d_0 = 1$, and $d_1 = \beta\rho^2 > 0$, where $d_i = 1 - 2q_i\sigma_x^2$. Thus the conditional moments evaluated above are finite for both choices. The residuals are the positive constants

$$\bar{B}_i = \exp \left(sa + \frac{s^2\bar{\sigma}^2}{2} - \frac{1}{2} \log d_i + \frac{\sigma_x^2 b_i^2}{2d_i} \right), \quad i = 0, 1.$$

Finally, set $Z_t^{(i)} = C_t \exp((q_i X_t^2 + b_i X_t)/s)$. By (16), the separable utilities are

$$U_t^{(i)} = \bar{B}_i^{\beta/[s(1-\beta)]} Z_t^{(i)} = C_t \exp \left(A_i + \frac{q_i X_t^2 + b_i X_t}{s} \right), \quad i = 0, 1,$$

where $A_i = \beta \log \bar{B}_i / [s(1 - \beta)]$. It remains to check that $C, Z^{(i)} \in \mathcal{L}_{++}^1(\delta)$.

A sufficient condition for $C, Z^{(i)} \in \mathcal{L}_{++}^1(\delta)$. Let

$$R := \exp \left(a + \frac{\bar{\sigma}^2}{2} + \frac{\sigma_x^2}{2(1 - \rho)^2} \right).$$

Now iterating the state equation gives $X_k = \rho^k X_0 + \sigma_x \sum_{j=1}^k \rho^{k-j} \varepsilon_j^x$. Substituting this expression into $\log(C_t/C_0) = ta + \sum_{k=0}^{t-1} X_k + \bar{\sigma} \sum_{j=1}^t Y_j$ and summing up the geometric series we obtain for $t \geq 1$

$$\begin{aligned} \log \frac{C_t}{C_0} &= ta + \frac{1 - \rho^t}{1 - \rho} X_0 + \bar{\sigma} \sum_{j=1}^t Y_j \\ &\quad + \sigma_x \sum_{j=1}^{t-1} \frac{1 - \rho^{t-j}}{1 - \rho} \varepsilon_j^x. \end{aligned}$$

As the shocks are independent standard normal random variables, we obtain

$$\begin{aligned}\mathbb{E}[C_t] &= C_0 \exp \left\{ t \left(a + \frac{\bar{\sigma}^2}{2} \right) + \frac{1 - \rho^t}{1 - \rho} X_0 + \frac{\sigma_x^2}{2} \sum_{j=1}^{t-1} \left(\frac{1 - \rho^{t-j}}{1 - \rho} \right)^2 \right\} \\ &\leq C_0 \exp \left(\frac{|X_0|}{1 - \rho} \right) R^t.\end{aligned}$$

This inequality uses the inequality $0 \leq (1 - \rho^k)/(1 - \rho) \leq 1/(1 - \rho)$ and the fact that the sum has at most t terms. Consequently, whenever $0 < \delta < \min\{1, R^{-1}\}$,

$$\sum_{t=0}^{\infty} \delta^t \mathbb{E}[C_t] \leq C_0 \exp \left(\frac{|X_0|}{1 - \rho} \right) \sum_{t=0}^{\infty} (\delta R)^t = \frac{C_0 \exp(|X_0|/(1 - \rho))}{1 - \delta R} < \infty.$$

It follows that $\delta R < 1$ implies $C \in \mathcal{L}_{++}^1(\delta)$. Incidentally, quick calculations also reveal that $\delta R < 1$ is a necessary condition for $C \in \mathcal{L}_{++}^1(\delta)$.

Finally, observe that since $0 < b_0/s < 1/(1 - \rho)$, the Gaussian AR(1) specification gives, for sufficiently large n and some constant $K > 0$,

$$Z_t^{(0)} \leq C_t + K \mathbb{E}_t[C_{t+n}].$$

Completing the square also gives $Z_t^{(1)} \leq e^{-b_1^2/(4q_1 s)} C_t$. Taking expectations and summing with weights δ^t therefore shows that admissibility of C implies admissibility of both generators and their constant multiples $U^{(i)}$.

Bansal–Yaron Case I calibration. Under (39) we have

$$\begin{aligned}\sigma_x &= 0.044 \times 0.0078 = 0.0003432, \\ q_0 &= 0, & b_0 &= \frac{0.998(-9)}{1 - 0.998(0.979)}, \\ q_1 &= \frac{1 - 0.998(0.979)^2}{2(0.0003432)^2}, & b_1 &= \frac{0.998(-9)(0.979)}{0.979 - 1}.\end{aligned}$$

Using these exact expressions in the formula for A_i above gives

$$\begin{aligned}A_0 &\simeq 0.1121, & b_0/s &\simeq 43.47, & q_0/s &= 0, \\ A_1 &\simeq -1.219, & b_1/s &\simeq -46.53, & q_1/s &\simeq -2.051 \times 10^4.\end{aligned}$$

Substituting these values into the formula found above we obtain the two recursive utilities

$$\begin{aligned}U_t^{(0)} &\simeq C_t \exp(0.1121 + 43.47 X_t), \\ U_t^{(1)} &\simeq C_t \exp(-1.219 - 46.53 X_t - (2.051 \times 10^4) X_t^2).\end{aligned}$$

For this calibration, the consumption admissibility condition is $0 < \delta < R^{-1}$, where

$$R^{-1} = \exp\left(-0.0015 - \frac{0.0078^2}{2} - \frac{0.0003432^2}{2(1 - 0.979)^2}\right) \simeq 0.9983.$$

Under the calibration in (39), subtracting the exponents of the two utilities gives

$$\log \frac{U_t^{(1)}}{U_t^{(0)}} \simeq -1.331 - 90.00X_t - (2.051 \times 10^4)X_t^2.$$

This quadratic is maximized at

$$X_t \simeq -\frac{90.00}{2(2.051 \times 10^4)} \simeq -0.002194,$$

where its value is

$$-1.331 + \frac{90.00^2}{4(2.051 \times 10^4)} \simeq -1.232 < 0.$$

Hence, we obtain that $U_t^{(1)} < U_t^{(0)}$ as desired.