Restricted Dynamic Consistency*

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Abstract

Dynamic consistency is a key behavioral property in dynamic models, enabling tractability by means of dynamic programming methods. However, it is a behavioral property that is often violated in experiments. This paper shows that dynamic consistency can be relaxed to hold over a much smaller domain of consumption programs. Nonetheless, this domain can still be sufficiently rich for practical applications. To illustrate, I provide examples of domains that are rich enough to separate risk aversion from intertemporal substitution. As an application, I introduce a new model of dynamic preferences, the Epstein-Zin-Selden-Stux preferences. These preferences are recursive only within a restricted domain. In contrast with standard recursive preferences, this weaker notion of dynamic consistency allows for indifference to the timing of resolution of uncertainty.

Keywords: Recursive utility, dynamic consistency, intertemporal substitution, risk aversion.

JEL classification: C61, D81.

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1 Introduction

Dynamic consistency is a key property for dynamic economic models. It implies that utilities satisfy a set of recursive conditions, which permits solving models by means of standard dynamic programming techniques. Moreover, this recursive formulation permits disentangling risk aversion from intertemporal substitution (e.g., see Chew and Epstein 1991). Because of these facts, dynamic preferences which satisfy dynamic consistency are of central importance in many economic settings, from models of consumption-based asset pricing (Epstein and Zin 1989, Epstein and Zin 1991) to optimal fiscal policy (Karantounias 2018).

At the same time, dynamic consistency is often violated in experiments. Indeed, behavioral properties related to dynamic consistency such as stationarity are often violated in experiments (Green et al. 1994; Kirby and Herrnstein 1995). Furthermore, there is direct evidence of empirical violations of dynamic consistency, such as the findings of Cubitt et al. (1998). Even when accounting for choice errors, both Busemeyer et al. (2000) and Johnson and Busemeyer (2001) provided support for these findings by identifying statistically significant violations of dynamic consistency. In contrast, other axioms of dynamic preferences, like consequentialism, have more favorable experimental support, as noted by Busemeyer et al. (2000). Halevy (2015) finds that around half of subjects in his experiments are dynamically inconsistent.

In this paper, I show that the axiom of dynamic consistency can be weakened so that it does not apply to all consumption programs. I consider preferences defined over the set of all conceivable consumption programs. However, these preferences satisfy the axiom of dynamic consistency only over a strict subset of consumption programs, whose only requirement is to be topologically connected. To illustrate the practical implications of this approach, I provide examples of consumption domains that are not only topologically connected but also *relevant* in applications.

The leading example of consumption domain I consider is the set of those program whose uncertainty about consumption resolves gradually (see Section 3). For example, this assumption excludes consumption programs whose uncertainty resolves after one period. To illustrate, in consumption-savings applications, consumption at every period is a function of income whose uncertainty resolves gradually. In consumption-based asset pricing models, consumption equals the the dividend process, whose uncertainty resolves gradually.

I show that despite this *restricted* notion of the dynamic consistency axiom, a recursive representation can still be obtained that applies over the restricted consumption domain (Theorem 1). In addition to being of relevance in applications, I show that consumption domains of this kind, nevertheless, are rich enough to allow for disentangling risk aversion from intertemporal substitution (Proposition 2). As a consequence, one can separate risk aversion from intertemporal substitution under weaker assumptions, while at the time considering a consumption domain that is relevant in applications.

To illustrate the implications of these results, I introduce a model that "merges" the Epstein-Zin preferences (Epstein and Zin 1989) with the dynamic ordinal certainty equivalent (DOCE) model of Selden and Stux (1978) (see also Selden 1978), an alternative approach to Epstein-Zin preferences to separate risk aversion from intertemporal substitution. Unlike Epstein-Zin preferences, DOCE preferences are characterized by being indifferent to the timing of resolution of uncertainty (see the discussion in Kubler et al. 2019a). I refer to this novel "hybrid" preferences as Epstein-Zin-Selden-Stux (EZSS).

EZSS preferences are represented by a utility function which is given by the standard Epstein-Zin formulation for consumption programs on the restricted domain of consumption, while for consumption programs not in this domain it takes the DOCE formulation. Notably, these preferences are therefore indifferent to the timing of resolution of uncertainty. In fact, this indifference is used to characterize EZSS preferences (see Theorem 2).

This new model of preferences can be used to explain recent puzzles concerning recursive preferences that have emerged in the literature. These puzzles are related to preferences for the timing of resolution of uncertainty. A recent literature has questioned the implications of the Epstein-Zin model. For example, Epstein et al. (2014) argue that standard parameter assumptions in finance and macroeconomic applications of the Epstein-Zin model can imply that consumers would pay implausibly large premia for early resolution of consumption risk. However, in an experiment, Meissner and Pfeiffer (2022) find that a sizable share of subjects are indifferent to the timing of resolution of uncertainty. EZSS preferences satisfy the assumptions of recursive utility over domains relevant in applications, but at the same time are indifferent to early resolution of uncertainty. Therefore, these preferences can reconcile these inconsistencies.

I consider a formal setting of *uncertainty* (unlike a setting of *risk* such as in Kreps and Porteus (1978) or Epstein and Zin (1989)). This setting permits considering recursive models of ambiguity aversion, which are well known to be relevant in applications (e.g see Ju and Miao (2012) or Collard et al. (2018)).

Related literature. The theoretical paper closest to the present one is Kubler et al. (2019a). They consider a consumption-portfolio optimization problem, and derive necessary and sufficient conditions DOCE preferences of Selden (1978) satisfy the properties of time consistency, the separation of time and risk preferences and the ability to accommodate an indifference to the timing of when risk is resolved. Johnsen and Donaldson (1985) provide an axiomatic representation of recursive preferences in a setting with two periods. Hammond (1989) shows that the work Johnsen and Donaldson contains a hidden assumption which restricts the domain of consumption. Epstein and Le Breton (1993) demonstrate that dynamic consistency implies expected utility. In the present paper, the filtration which describes the evolution of information is taken to be fixed, therefore their results do not apply. Notice that a fixed filtration is a standard assumption in applications. Sarin and Wakker (1998) and Epstein and Schneider (2003b) characterize recursive multiple prior preferences. Siniscalchi (2011) focuses on dynamically inconsistent preferences. Bommier et al. (2017) and Marinacci et al. (2023) show that the assumption of monotonicity has strong restrictions on recursive preferences.

2 Preliminaries

2.1 Framework

Time is discrete and varies over a finite horizon $t \in \{0, 1, \ldots, T\} \equiv \mathcal{T}$. The information structure is described by a filtered space $(\Omega, \{\mathcal{G}_t\}_{t \in T})$ where Ω is an arbitrary set of states of the world and $\mathcal{G} = \{\mathcal{G}_t\}_{t \in T}$ is a sequence of σ-algebras such that $\mathcal{G}_0 = \{\Omega, \emptyset\}$ that satisfy $\mathcal{G}_t \subset \mathcal{G}_{t+1}$ for $t = 0, \ldots, T-1$.[1] For simplicity, assume that every algebra $\mathcal{G}_t, t \in T$, is generated by a finite partition, where $\mathcal{G}_t(\omega)$ denotes the element of the partition containing $\omega \in \Omega$.

Let X denote the set of outcomes, which is assumed to be a convex subset of

[1] See Stokey and Lucas (1989) for canonical interpretations of this setting in terms of shocks/observations.

$\mathbb{R}^n$. The main cases we are interested in are $X = \mathbb{R}_+$ and $X = [1, \infty)$. An act or consumption program is an X-valued, $\mathcal{G}$-adapted process, that is, a sequence $(f_t)_{t \in \mathcal{T}}$ such that $f_t : \Omega \to X$ is $\mathcal{G}_t$ measurable for every $t \in \mathcal{T}$. $\mathcal{F}$ is the set of all consumption programs. $\mathcal{D}$ is the set of deterministic consumption programs, $d = (d_0, d_1, \ldots, d_n) \in \mathcal{D}$ if and only if d_t is measurable w.r.t. $\mathcal{G}_0$ for all t. Since each $\mathcal{G}_t$ is finitely generated, then set of all $\mathcal{G}_t$-measurable acts can be endowed with the product topology. It follows that we can endow $\mathcal{F}$ with the product topology.[2] The assumptions of finiteness are not necessary but avoid the need to formally establish the existence of each recursive utility model considered in this paper.[3]

Given a measurable space (S, Σ) and $K \subseteq \mathbb{R}$, let $B_0(\Sigma, K)$ denote the set of simple Σ measurable function with range contained in K. A function $I : B_0(\Sigma, K) \to \mathbb{R}$ (i) continuous if it continuous in the sup-norm topology (ii) monotone if $\xi(s) \geq \xi'(s)$ for every $s \in S$ implies $I(\xi) \geq I(\xi')$ (iii) strictly monotone if it is monotone and $\xi(s) \geq \xi'(s)$ for every $s \in S$ with one strict inequality implies $I(\xi) > I(\xi')$ (iv) normalized if $I(x) = x$ for every $x \in \mathbb{R}$ (where x denotes the constant function $x1_\Omega$). Call I a *certainty equivalent* if it is continuous, strictly monotone and normalized. Given a probability measure P defined on (S, Σ), an expected utility functional is given by $\mathbb{E}_P \xi = \int \xi(s) \mathrm{d}P(s)$.

The primitives of interest are a family of $\mathcal{G}$-adapted weak orders (complete and transitive relations) $\{\succeq_{t,\omega}\}_{(t,\omega) \in \mathcal{T} \times \Omega}$ on $\mathcal{F}$.[4] Let $\succeq_0$ denote the preference at time zero. For brevity, I typically denote the collection of preferences $\{\succeq_{t,\omega}\}_{(t,\omega) \in \mathcal{T} \times \Omega}$ with $\succeq_{t,\omega}$ only. Say that a collection of real-valued functions $(V_t(\omega, \cdot))_{t,\omega}$ defined on $\mathcal{R}$ such that $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{F}$ represents $\succeq_{t,\omega}$ over $\mathcal{R}$ if for every $h, h' \in \mathcal{R}$

$$V_t(\omega, h) \geq V_t(\omega, h') \iff h \succeq_{t,\omega} h' \text{ for every } (t, \omega).$$

2.2 Recursive preferences

I consider a general notion of recursive representation of preferences.

[2]Denoting with $|\mathcal{G}_t|$ the number of elements of the partition that generates $\mathcal{G}_t$, the set

$$\{f : \Omega \to X : f \text{ is measurable w.r.t. } \mathcal{G}_t\},$$

can be identified with a subset of $\mathbb{R}^{|\mathcal{G}_t|}$, and therefore $\mathcal{F}$ can be endowed with the product topology.

[3]See Marinacci and Montrucchio (2010) for a thorough treatment of the topic.

[4]By $\mathcal{G}$-adapted I mean that $\succeq_{t,\omega} = \succeq_{t,\omega^*}$ whenever $\mathcal{G}_t(\omega) = \mathcal{G}_t(\omega^*)$.

Definition 1. *Let $\mathcal{R}$ satisfy $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{F}$. Preferences $\succeq_{t,\omega}$ admit a recursive representation over $\mathcal{R}$ if and only if there exist $(V_t(\omega, \cdot))_{t,\omega}$ that represent $\succeq_{t,\omega}$ over $\mathcal{R}$ satisfying the recursive relation $V_T(\omega, h) = u(h_T(\omega))$ for some continuous $u : X \to \mathbb{R}$ that satisfies $u(z) = 0$ for some $z \in X$ and for $t < T$,*

$$V_t(\omega, h) = W\left(h_t(\omega), I_{t,\omega}\left(V_{t+1}(\cdot, h)\right)\right) \text{ for every } h \in \mathcal{R}, \tag{1}$$

where $V_t(\omega, \mathcal{R}) = V_t(\omega', \mathcal{R}) \equiv V_t$, each

$$I_{t,\omega} : \{\xi \in B_0(\mathcal{G}_{t+1}, V_{t+1}) : \xi = V_{t+1}(\cdot, f), f \in \mathcal{R}\} \to \mathbb{R},$$

is a certainty equivalent satisfying $I_{t,\omega} = I_{t,\omega^*}$ when $\mathcal{G}_t(\omega) = \mathcal{G}_t(\omega^*)$ and $W : X \times \cup_{\tau \geq t+1} V_\tau \to \mathbb{R}$ is a time aggregator that is continuous and strictly increasing in the second variable that satisfies $W(x, u(z)) = u(x)$.

A general recursive representation of $\succeq_{t,\omega}$ can be identified by its parameters $(W, u, (I_{t,\omega})_{t,\omega})$. Below I describe the most common types of specifications.

- Recursive discounted expected utility (RDEU) preferences, where $W(x, y) = u(x) + \beta y$, $\beta \in (0, 1)$ and $I_{t,\omega}(\xi) = \mathbb{E}_{P_{t,\omega}}\xi$ with each $P_{t,\omega}$ being a probability on $(\Omega, \mathcal{G}_{t+1})$.

- Recursive second-order expected utility preferences, where $W(x, y) = u(x) + \beta y$, $\beta \in (0, 1)$ and $I(\xi) = \phi^{-1}\left(\mathbb{E}_{P_{t,\omega}}\phi(\xi)\right)$ for some strictly increasing and concave function $\phi : u(X) \to \mathbb{R}$ and each $P_{t,\omega}$ is a probability on Ω. Such a class of recursive preferences offers a simple separation between risk aversion and intertemporal substitution. The most important instance of such preferences are Epstein-Zin preferences (EZ) Epstein and Zin (1989), which are given by[5]

$$u(x) = \begin{cases} \frac{x^\rho}{\rho} & 0 \neq \rho < 1, \\ \log(x) & \rho = 0, \end{cases}$$

  and

$$\phi(x) = \begin{cases} \frac{\rho}{\alpha} x^{\frac{\alpha}{\rho}} & 0 \neq \alpha < 1, 0 \neq \rho < 1, \\ \frac{1}{\alpha} \exp \alpha x & 0 \neq \alpha < 1, \rho = 0. \end{cases}$$

[5] Epstein and Zin (1989) consider a more general class of certainty equivalents, but for simplicity I focus on the case of expected utility.

- Recursive maxmin expected utility (RMEU) preferences, see Epstein and Wang (1994), Epstein and Schneider (2003b) where $I(\xi) = \min_{p \in C_{t,\omega}} \mathbb{E}_p \xi$, with each set $C_{t,\omega}$ being convex and weak * -closed set of probabilities with full support on $(\Omega, \mathcal{G}_{t+1})$.

3 Relevant domains of consumption

In the standard dynamic consumption problem under uncertainty, uncertainty about consumption resolves gradually. Therefore, programs that feature one-shot resolution of uncertainty are not needed to solve this kind of problem. To illustrate, in the consumption-savings applications, consumption c_t at every period t is a non-trivial function of income y_t, and uncertainty about income resolves gradually. In consumption-based asset pricing models, in equilibrium one has $c_t = d_t$ where $(d_t)_t$ is the dividend process, whose uncertainty resolves gradually.

For this reason, I suggest one should not take $\mathcal{F}$ as a domain of choice, but rather a strict subset of it, a *relevant* domain in applications. A consumption program $f \in \mathcal{F}$ involves early resolution if for some $t \geq 1$, f_t is measurable w.r.t. $\mathcal{G}_\tau$ for some $\tau < t$. In words, this means that time t consumption is known at the earlier period τ.

Definition 2 (Domain of gradual resolution)**.** *For every $t \in T$, let $\mathcal{F}_t$ denote the set given by*

$$\mathcal{F}_t = \{f \in \mathcal{F} : f_t \text{ is } \mathcal{G}_\tau\text{-measurable for some } \tau < t \implies f \in \mathcal{D}\}.$$

Define the relevant domain to be $GR = \cap_{t=1}^{T} \mathcal{F}_t$.

In words, this means that a consumption program in GR either involves no early resolution or it is deterministic. To illustrate, Figure 1 provides an example of consumption programs (assuming $x \neq y$) that resolve early (bottom) and gradually (top). Therefore, only the top consumption program belongs to GR.

Another important example of consumption domain relevant in applications is that of consumption programs that are independently distributed over time.

Definition 3. *The set of independent consumption programs is given by*

$$IND = \{h \in \mathcal{F} : \text{ there exist } (f_t)_t \text{ with } f_t \in X^S \text{ such that } h_t(s_1, \ldots, s_{t-1}, \cdot) = f_t(\cdot), \\ \text{and if for some } t', f_{t'} \text{ is constant } \implies h \in \mathcal{D}\}.$$

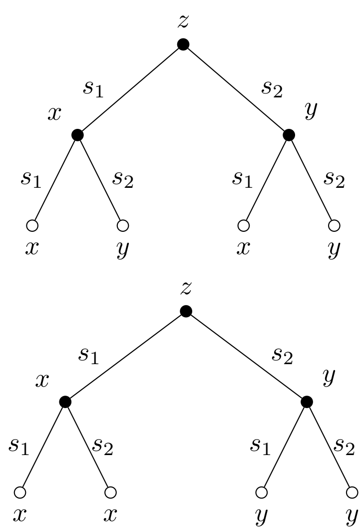

Figure 1: Gradual resolution vs early resolution.

In words, this set contains consumption programs that are "independent" (h_t does not depend on $(s_1, \ldots, s_{t-1})$ but not necessarily "identically distributed" (h_t depends on a function $f_t : S \to X$ which is not identical over time). Observe that GR and IND are also mathematically rich enough to axiomatize a general recursive representation and disentangle a general notion of risk aversion from intertemporal substitution. These results, which I discuss in the next section, rely on the following proposition.

Proposition 1. *The consumption domains GR and IND are separable and connected metric spaces.*

4 Recursive utility over the relevant domain

Let $\mathcal{R}$ satisfy $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{F}$. I show that if $\mathcal{R}$ is topologically connected then one can obtain an aximatic representation of recursive preferences. The first axiom is a standard continuity requirement.

Axiom 1 (Continuity)**.** For every $h \in \mathcal{R}$ the sets

$$\{f \in \mathcal{F} : f \succeq_{t,\omega} h\},$$

and

$$\{f \in \mathcal{F} : h \succeq_{t,\omega} f\},$$

are closed.[6]

Given $\tau \in T$, $x, y, z \in X$, and $d \in \mathcal{D}$, $(d_{-t-1}, y, x_{T-t'}, z_{t-t'})$ denotes the deterministic consumption stream that pays d_τ at times $\tau = 0, \dots, t-1$, y at time t, x at times $\tau = t+1, \dots, T+t-t'$ and z at times $\tau = T+t-t'+1, \dots, T$. The next axiom, stationarity, requires preference over deterministic programs to be independent of a time delay.

Axiom 2 (Stationarity)**.** There exist $z \in X$ such that for every $t \leq t', \omega, \omega' \in \Omega$, $d \in \mathcal{D}$, $y, \bar{y}, x, \bar{x} \in X$

$$(d_{-t-1}, y, x_{T-t'}, z_{t-t'}) \succeq_{t,\omega} (d_{-t-1}, \bar{y}, \bar{x}_{T-t'}, z_{t-t'}) \iff (d_{-t'-1}, y, x_{T-t'}) \succeq_{t',\omega'} (d_{-t'-1}, \bar{y}, \bar{x}_{T-t'}).$$

The next axiom, which I refer to as consequentialism, requires that the decision maker at a note (t, ω) does not care about (i) what an act pays on unrealized events nor (ii) what it pays at earlier time periods.

Axiom 3 (Consequentialism)**.** For all $t \in T$ and $\omega \in \Omega$, and all acts $f, g \in \mathcal{F}$, if $f_k(\omega') = g_k(\omega')$ for all $k \geq t$ and all $\omega' \in \mathcal{G}_t(\omega)$, then $f \sim_{t,\omega} g$.

Observe that the above axiom implies that the ranking of a program $f \in \mathcal{R}$ by $\succeq_{t,\omega}$ depends only on $(f_t(\omega), f_{t+1}, \dots, f_T)$.

Finally, the last axiom excludes preference reversals as new information arrives.

Axiom 4 (Restricted Dynamic Consistency)**.** For all $t \in T$, and $\omega \in \Omega$, and programs $f, g \in \mathcal{R}$ that yield identical outcomes up to and including period t, if $f \succeq_{t+1,\omega'} g$ for all $\omega' \in \mathcal{G}_t(\omega)$, then $f \succeq_{t,\omega} g$ and if $f \succ_{t+1,\omega'} g$ for some $\omega' \in \mathcal{G}_t(\omega)$, then $f \succ_{t,\omega} g$.[7]

[6] Recall that $\mathcal{F}$ is endowed by the product topology, and that therefore $\mathcal{R}$ can be endowed with the relative topology

[7] It is possible to consider a weaker axiom notion of dynamic incosistency, which would result in a certainty equivalent that $I_{t,\omega}$ need not be strictly monotone. This could be done by defining appropriately the notion of a $\succeq_{t,\omega}$-nonnull event. Then one can require that if $f \succ_{t+1,\omega'} g$ for every ω' in a $\succeq_{t,\omega}$ -nonnull event, then $f \succ_{t,\omega} g$. I chose to present the stronger representation as DC is easier to state.

Observe that this version of dynamic consistency is *restricted* to hold only for the relevant domain of consumption $\mathcal{R}$, and need not hold on the entire domain $\mathcal{F}$. The next representation theorem characterizes recursive utility under very general conditions (cf. Kreps and Porteus (1978), Johnsen and Donaldson (1985), Chew and Epstein (1991), Skiadas (1998), Wang (2003), Hayashi (2005), Bommier et al. (2017)), allowing for both changing beliefs and ambiguity sensitive preferences. The only loss of generality is constituted by the exclusion of an infinite horizon, which however can be overcome by means of appropriate technical conditions.

Theorem 1 (Recursive representation)**.** *Assume that $\mathcal{R}$ is connected and satisfies $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{F}$. Preferences $\succeq_{t,\omega}$ satisfy axioms 1-4 if and only if they admit a general recursive representation over $\mathcal{R}$.*

Proof. See the appendix. □

At a technical level, the main difficulty introduced by weakening the completeness axiom is related to showing that $\mathcal{R} \subseteq \mathcal{F}$ is rich enough to construct a representation. In the appendix (see Lemma and 1 and Remark 4) I show that GR and IND are connected. Hence, Theorem 1 shows that not only is gradual resolution of uncertainty the relevant case in applications, but that also it is enough to provide a recursive representation of preferences. These domains of consumption are silent about preferences for early or late resolution of uncertainty, the main motivation of Kreps and Porteus (1978).

I close with a few remarks.

Remark 1. One could wonder whether given a recursive representation $(W, u, (I_{t,\omega})_{t,\omega})$ on $\mathcal{R}$ the only "reasonable" extension to $\mathcal{F}$ is given by the straightforward extension of $(W, u, (I_{t,\omega})_{t,\omega})$ to $\mathcal{F}$. In Section 4.2, I show that one can extend preferences in a different fashion. Specifically, I introduce preferences that have an Epstein-Zin representation on $\mathcal{R}$ but on $\mathcal{F} \setminus \mathcal{R}$ admit the representation introduced by Selden and Stux (1978) and Selden (1978). Notably, such a "hybrid" model features indifference to timing of resolution of uncertainty.

Remark 2. The theorem makes no reference to uniqueness of the representation. Uniqueness can be achieved by adding further conditions that imply uniqueness of $u : X \to \mathbb{R}$. For example, one can assume that X is the set of lotteries over a finite set Z and obtain uniqueness of u by means of specific axioms such as independence.

4.1 Separating intertemporal substitution from attitudes toward uncertainty

A simple yet important implication of Theorem 1 is that to separate risk aversion from the intertemporal rate substitution it is enough to observe only choices over a subset $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{F}$.

Comparative risk aversion can be defined in a similar fashion as in Epstein and Zin (1989) (pp. 949-950) and Chew and Epstein (1991) (Theorem 3.2). For any $f \in \mathcal{R}$, (t, ω) and $d \in \mathcal{D}$, denote with $(f_t(\omega), d_{T-t})$ the consumption stream that pays $f_t(\omega)$ at time t and d_τ at times $\tau = t+1, \ldots, T$.

Definition 4. *$\succeq^1_{t,\omega}$ is more risk averse than $\succeq^2_{t,\omega}$ if for every $f \in \mathcal{R}$, $d \in \mathcal{D}$ and (t, ω) with $t < T$*

$$(f_t(\omega), d_{T-t}) \succeq^2_{t,\omega} (f_t(\omega), f_{t+1}, \ldots, f_T) \implies (f_t(\omega), d_{T-t}) \succeq^1_{t,\omega} (f_t(\omega), f_{t+1}, \ldots, f_T),$$

and

$$(f_t(\omega), d_{T-t}) \succ^2_{t,\omega} (f_t(\omega), f_{t+1}, \ldots, f_T) \implies (f_t(\omega), d_{T-t}) \succ^1_{t,\omega} (f_t(\omega), f_{t+1}, \ldots, f_T).$$

We obtain the following comparative statics result.

Proposition 2. *Consider preferences $\succeq^i_{t,\omega}$, $i = 1, 2$ that admit the representation in (1). $\succeq^1_{t,\omega}$ is more risk averse than $\succeq^2_{t,\omega}$ if and only if they admit recursive representations $(W^i, u^i, (I^i_{t,\omega})_{t,\omega})$, $i = 1, 2$ such that $u^1 = u^2$, $W^1 = W^2$ and $I^1_{t,\omega}(\xi) \leq I^2_{t,\omega}(\xi)$ for every $\xi \in \{\xi \in B_0(\mathcal{G}_{t+1}, V_{t+1}) : \xi = V_{t+1}(\cdot, f), f \in \mathcal{R}\}$ and every (t, ω).*

Remark 3. Observe that if $\mathcal{R} = \mathcal{D}$ then the "only if" part of the statement is trivially true since $I^i_{t,\omega}$ are defined on deterministic prospects so that $I^1_{t,\omega} = I^2_{t,\omega}$. More in general, this will be true whenever

$$\{\xi \in B_0(\mathcal{G}_{t+1}, V_{t+1}) : \xi = V_{t+1}(\cdot, f), f \in \mathcal{R}\} = \{\xi \in B_0(\mathcal{G}_{t+1}, V_{t+1}) : \xi = V_{t+1}(d), d \in \mathcal{D}\}. \tag{2}$$

Now consider the special case of EZ preferences. Assume that $\succeq^i_{t,\omega}$ are represented by

$$V^i_t(\omega, h) = \frac{h_t(\omega)^{\rho_i}}{\rho_i} + \beta_i (\mathbb{E}_{P_{t,\omega}}(V^i_{t+1}(\cdot, h)^{\frac{\alpha_i}{\rho_i}})^{\frac{\rho_i}{\alpha_i}}, \tag{3}$$

for $0 \neq \rho < 1$ and that (2) does not hold. In this case we obtain the following.

Corollary 1. *$(\succeq_{t,\omega})^1_{t,\omega}$ is more risk averse than $(\succeq_{t,\omega})^2_{t,\omega}$ if and only if $\beta_1 = \beta_2$, $\rho_1 = \rho_2$ and $\alpha_1 \leq \alpha_2$.*

Proof. The result follows immediately by the previous proposition upon observing that $I(\xi) = (\mathbb{E}_P \xi^\alpha)^{\frac{1}{\alpha}}$ is increasing in α (see Theorem 16, Hardy et al. (1952)). □

This result establishes that domains of consumption such as $\mathcal{R} = GR$ or $\mathcal{R} = IND$ are rich enough to disentangle risk aversion from intertemporal substitution, since (2) does not hold in these cases.

4.2 The Epstein-Zin-Selden-Stux model

Within the consumption domains GR and IND, uncertainty only resolves gradually. Therefore, within these domains one cannot infer preferences for early or late resolution of uncertainty (Kreps and Porteus 1978). Here I present an example of preferences $\mathcal{F}$ that are recursive over $\mathcal{R} \subseteq \mathcal{F}$, but at the same time are indifferent to the timing of resolution. In this way, this new model of preferences is able to addresses puzzles in the literature, particularly concerning how consumers value the timing of uncertainty resolution. Epstein et al. (2014) argue that standard parameter assumptions the Epstein-Zin model implies consumers would pay implausibly large timing premia for early resolution of consumption risk, while Meissner and Pfeiffer (2022) find that a consistent subjects in their experiment are indifferent to the timing of resolution of uncertainty.

The new model I introduce merges EZ preferences with DOCE preferences. Axiomatized in Selden (1978) and Selden and Stux (1978), DOCE preferences replace risky consumption in each period by certainty equivalents with respect to a utility function $v(\cdot)$ and evaluate the resulting sequence of certainty equivalents with discounted utility with respect to a utility function $u(\cdot)$. These preferences an alternative to EZ preferences to disentangle risk aversion from intertemporal substitution. In contrast with recursive preferences, DOCE preferences are neutral to the timing of resolution of uncertainty (see Selden and Stux (1978) and Kubler et al. (2019b)). Hall (1985), Zin et al. (1987), Attanasio and Weber (1989) and Kubler et al. (2019b) have studied applications of different variations of such preferences.

In order to consider the behavioral property of indifference to the timing of resolution of uncertainty, I consider a setting of IID (Independently and Indistinguishably

Distributed) ambiguity (see Epstein and Schneider (2003a) and Strzalecki (2013)). Specifically, such an assumption requires that that $\Omega = S^T$ with $T \geq 2$, where (S, Σ) is a finite measurable space. Moreover, $\Sigma = 2^S$ and let $\mathcal{G}_t = \Sigma^t \times \{\emptyset, S\}^{T-t}$. In words, this means that at time t one knows the realization of $(\omega, t) = (s_1, \ldots, s_t) := s^t$, but is ignorant about the future. More precisely, observe that in this case we have $\mathcal{G}_t((s_1, \ldots, s_T)) = \{s_1\} \times \ldots \times \{s_t\} \times \{\emptyset, S\}^{T-t}$. Finally, assume that $X = [0, \infty)$

In this setting, the Epstein-Zin-Selden-Stux preferences (EZSS) are defined as follows.

Definition 5. *Preferences $\succeq_{t,\omega}$ admit an EZSS representation if they are represented over $\mathcal{F}$ by $(V_t(s^t, \cdot))_{s^t}$ which for some $0 \neq \alpha < 1, 0 \neq \rho < 1$ satisfies for every $h \in \mathcal{F} \setminus \mathcal{R}$*

$$V_t(s^t, h) = \frac{h_t(s_1, \ldots, s_t)^\rho}{\rho} + \sum_{j=1}^{T-t} \beta^j \frac{1}{\rho} \left[\mathbb{E}_{\prod_{\tau=1}^{j} P(s_{t+j})} h_{t+j}^\alpha (s^t, \cdot) \right]^{\frac{\rho}{\alpha}}, \tag{4}$$

and for every $h \in \mathcal{R}$

$$V_t(s^t, h) = \frac{h_t(s^t)^\rho}{\rho} + \beta(\mathbb{E}_P (V_{t+1}(\cdot, h)^{\frac{\alpha}{\rho}})^{\frac{\rho}{\alpha}}. \tag{5}$$

Preferences that admit an EZSS representation satisfy dynamic consistency only on the relevant domain $\mathcal{R}$. At the same time, as I am going to show, they satisfy indifference to the timing of resolution of uncertainty.

Consider preferences $\succeq_{t,\omega}^{EZ}$ that have an EZ representation (see Section 2.2) on $\mathcal{F}$ with parameters (α, ρ), $0 \neq \alpha < 1, 0 \neq \rho < 1$. This setting permits defining ranking consumption programs in terms of temporal resolution of uncertainty to define indifference to timing (see Strzalecki (2013)).

Definition 6. *Fix $t \leq T - 2$. Say that $h \in \mathcal{F}$ resolves earlier than $h' \in \mathcal{F}$ whenever there exist $f_{t+2}, \ldots, f_T \in X^S$ and $x_0, \ldots, x_{t+1} \in X$ such that $h_j = h'_j = x_j$ for all $j \leq t+1, h_j(s_1, \ldots, s_j) = f_j(s_{t+1})$ for $j \geq t+2$, and $h'_j(s_1, \ldots, s_j) = f_j(s_{t+2})$.*

The first axiom requires indifference toward the timing of resolution of uncertainty.

Axiom 5 (Indifference to timing)**.** Consider $h, h' \in \mathcal{F} \setminus \mathcal{R}$ such that for some $\bar{s}^t = (\bar{s}_1, \ldots, \bar{s}_t)$ with $1 \leq t \leq T - 2$ the consumption program

$$(h_0, h_1(\bar{s}_1), h_2(\bar{s}_1, \bar{s}_2), \ldots, h_t(\bar{s}^t), \ldots, h_T(\bar{s}^t, \cdot)),$$

resolves earlier than

$$(h'_0, h'_1(\bar{s}_1), h'_2(\bar{s}_1, \bar{s}_2), \ldots, h'_t(\bar{s}^t), \ldots, h'_T(\bar{s}^t, \cdot)),$$

and $h_\tau(s_1, \ldots, s_\tau) = h'_\tau(s_1, \ldots s_\tau)$ for every $(s_1, \ldots, s_\tau)$ with $\tau < t$ and $h_\tau(s_1, \ldots, s_\tau) = h'_\tau(s_1, \ldots s_\tau)$ for every $(s_1, \ldots s_\tau)$ such that $\tau \geq t$ and $(s_1, \ldots, s_t) \neq (\bar{s}_1, \ldots, \bar{s}_t)$. Then it holds that $h \sim_{t',\omega} h'$ for every $t' \leq t$.

In words, if h resolves earlier than h', but otherwise these programs are equivalent, one should be indifferent. The next axiom requires preferences to admit an EZ representation over $\mathcal{R}$, therefore implying axioms 1, 2 and 4.

Axiom 6 (Epstein-Zin over the relevant domain)**.** For every $h, h' \in \mathcal{R}$

$$h \succeq_{t,\omega} h' \iff h \succeq^{EZ}_{t,\omega} h'.$$

The next two axioms require that, for each period, risky consumption profiles are assessed through certainty equivalents, consistent with the Epstein-Zin preferences $\succeq^{EZ}_{t,\omega}$.

Axiom 7 (Consistency with Epstein-Zin)**.** Let $h, h' \in \mathcal{R}$ be such that there exist, t, $f, f' : S \to X$ and $(s_1, \ldots, s_{t-1})$ such that $h_t(s_1, \ldots, s_{t-1}, \cdot) = f(\cdot)$, $h'_t(s_1, \ldots, s_{t-1}, \cdot) = f'(\cdot)$, $h_\tau = h'_\tau = 0$ for all $\tau \neq t$ and $h_t(\bar{s}^t) = h'_\tau(\bar{s}^t) = 0$ whenever $(\bar{s}_1, \ldots, \bar{s}_{t-1}) \neq (s_1, \ldots, s_{t-1})$. Then

$$h \succeq_{t,\omega} h' \iff h \succeq^{EZ}_{t,\omega} h'.$$

Axiom 8 (Risk Independence)**.** Given any pair $h, h' \in \mathcal{F} \setminus \mathcal{R}$ which are identical except at the node $(s_1, \ldots, s_{t-1})$, then letting $\bar{h}_t(s_1 \ldots, s_{t-1}, \cdot) = h_t(s_1 \ldots, s_{t-1}, \cdot)$, $\bar{h}'_t(s_1 \ldots, s_{t-1}, \cdot) = h'_t(s_1 \ldots, s_{t-1}, \cdot)$ and $\bar{h}_t = \bar{h}'_t = 0$ otherwise,

$$\bar{h} \sim_{t,\omega} \bar{h}' \implies h \sim_{t,\omega} h'.$$

These axioms, along with the axiom of consequentialism (axiom 3), characterize EZSS preferences.

Theorem 2. *Assume that $\mathcal{R}$ is connected and satisfies $\mathcal{D} \subseteq \mathcal{R} \subseteq \mathcal{F}$. Preferences $\succeq_{t,\omega}$ satisfy axioms 3, 5-8 if and only if they admit an EZSS representation.*

This result shows that Epstein-Zin recursive utility can be compatible with indifference to the timing of resolution of uncertainty.

5 Concluding remarks

Models of recursive preferences play a central role in many applications in economics. However, they require strong assumption on behavior. One objection against the axiom of dynamic consistency is that it is unrealistic to assume it can hold even when the decision maker is confronted with unrealistic choice situations. I provided an axiomatization of recursive preferences based on much weaker assumptions than what is usually assumed. At the same time, these assumptions are still strong enough for recursive models to be applied. Moreover, this approach can be used to address existing empirical puzzles in the applied literature.

6 Appendix

6.1 Proof of Proposition 1

Proof. First observe that $\mathcal{F}$ is separable and therefore GR is separable since any subset of a separable metric space is separable. I now show that GR is path-connected. Take $h \in GR$ and $d \in \mathcal{D}$. Clearly if $h \in \mathcal{D}$ then the result follows by convexity of $\mathcal{D}$ (recall that X is convex). Assume $h \in GR \backslash \mathcal{D}$. Let $\{P_1^t, \ldots, P_{n^t}^t\}$ denote the partition of Ω that generates $\mathcal{G}_t$. I construct a continuous path $\iota : [0,1] \to GR$ that connects h to d. Since X is convex, for every t we just let $\iota_t(\alpha) = (1-\alpha)h_t + \alpha d_t$. Fix $t \geq 1$. Without loss of generality, assume that $n^t - n^{t-1} = 1$ and $P_{n^{t-1}}^{t-1} = P_{n^t}^t \cup P_{n^t-1}^t$. Let $\omega \in P_{n^t}^t$ and $\omega' \in P_{n^t-1}^t$. If $(1-\alpha)h_t(\omega) + \alpha d_t = (1-\alpha)h_t(\omega') + \alpha d_t$, we obtain a contradiction since $h_t(\omega) = h_t(\omega')$ but $h \in GR \setminus \mathcal{D}$. Therefore, $\iota_t(\alpha) \in GR$ for every α. It follows that we can connect via a path any $f \in GR$ to $d \in \mathcal{D}$. Hence, we can connect any $h, h' \in GR$ by a path. We conclude that GR is path-connected and therefore connected. $\square$

Remark 4. The set

$$IND = \{h \in \mathcal{F} : \text{ there exist } (f_t)_t \text{ with } f_t \in X^S \text{ such that } h_t(s_1, \ldots, s_{t-1}, \cdot) = f_t(\cdot), \\ \text{and if for some } t', f_{t'} \text{ is constant } \implies h \in \mathcal{D}\},$$

is easily seen to be connected by analogous arguments. Observe that such a domain is the natural extension to T periods of "certain $\times$ uncertain" consumption plans (e.g., see Selden (1978), Johnsen and Donaldson (1985)). Indeed, the two coincide when $T = 1$.

6.2 Proof of Theorem 1

Proof of Theorem 1. I first prove sufficiency of the axioms. First by continuity, consequentialism and since $\mathcal{R}$ is connected and separable (the consumption set is separable) by assumption, one can apply well known results from Debreu (1954) to show that there exist (sequentially) continuous functions $(V_t(\omega,\cdot))_{t,\omega}$ such that

$$V_t(\omega,h) = V_t(h_t(\omega), h_t, \ldots, h_T) \quad \text{for every } h \in \mathcal{R}.$$

Observe that by stationarity there exists a (sequentially) continuous function $u : X \to \mathbb{R}$ such that $V_T(\omega,h) = u(h_T(\omega))$ and $V_t(\omega,(x, z_{T-t-1})) = u(x)$ for every $\omega \in \Omega$ and $t < T$. Moreover, we can normalize $u(\cdot)$ from the stationarity axiom so that $u(z) = 0$.

I construct $I_{t,\omega} : B_0(\mathcal{G}_{t+1}, V_{t+1}(\omega,\mathcal{R})) \to \mathbb{R}$ as follows: for every h, by continuity, dynamic consistency, consequentialism, and stationarity we can construct $d_{\omega,t} = (d_{t+1},\ldots,d_T) \in X^{T-t}$ such that for any $\bar{d} \in \mathcal{D}$

$$h \sim_{t,\omega} (\bar{d}_{-t-1}, h_t(\omega), d_{\omega,t}) \in \mathcal{D}. \tag{6}$$

Observe that all acts in (6) belong to $\mathcal{R}$. In particular, $d_{\omega,t}$ can be constructed recursively as follows. Starting from $t = T-1$, observe that for any $\omega \in \Omega$, there exist $x, y \in X$ such that

$$V_{T-1}(h_{T-1}(\omega), x) \geq V_{T-1}(h_{T-1}(\omega), h_T) \geq V_{T-1}(h_{T-1}, y).$$

To see this, let $x = h_T(\bar{\omega})$ and $y = h_T(\underline{\omega})$, where $\bar{\omega} = \arg\max_\omega u(h_T(\omega))$ and $\underline{\omega} = \arg\min_\omega u(h_T(\omega))$. The statement follows by applying dynamic consistency. Therefore, by continuity and connectedness X we can find $d_{T-1,\omega} \in X$ such that $h \sim_{T-1,\omega} (\bar{d}_{-t-1}, h_T(\omega), d_{T-1,\omega})$. Now for any $t < T-1$ and ω, assume one has constructed $d_{t+1,\omega'}$ for every $\omega' \in \mathcal{G}_t(\omega)$. Let $\bar{d}_{t,\omega} = (h_{t+1}(\bar{\omega}), d_{t+1,\bar{\omega}})$ and $\underline{d}_{t,\omega} = (h_{t+1}(\underline{\omega}), d_{t+1,\underline{\omega}})$ where

$$\bar{\omega} = \arg\max_{\omega'} V(h_{t+1}(\omega), d_{t+1,\omega'}),$$

and

$$\underline{\omega} = \arg\min_{\omega'} V(h_{t+1}(\omega), d_{t+1,\omega'}),$$

Then by dynamic consistency and stationarity we have

$$V_t(h_t(\omega), \bar{d}_{t,\omega}) \geq V_t(\omega, h) \geq V_t(h_t(\omega), \underline{d}_{t,\omega}).$$

Again, by connectedness of X and continuity we can find $d_{t,\omega}$ such that (6) is verified.

Now observe that this implies that for each (t, ω), $t = 0, \dots, T$ and $\omega \in \Omega$ we have $V_t(\omega, \mathcal{R}) = V_t(\omega', \mathcal{R}) \equiv V_t$ (observe that $V_t \subseteq V_{t'}$ whenever $t' \leq t$). Define

$$I_{t,\omega} : B_0(\mathcal{G}_{t+1}, V_{t+1}) \to \mathbb{R},$$

by $I_{t,\omega}(\xi) = V_{t+1}(d_{\omega,t})$ and where $\xi(\omega) = V_{t+1}(\omega, h)$. Observe that $I_{t,\omega}$ is well defined by dynamic consistency.

I now claim that $I_{t,\omega}$ is strictly monotone, normalized and continuous. That $I_{t,\omega}$ is normalized follows by definition. Strict monotonicity follows by dynamic consistency. To prove continuity, assume that $\xi_n \to \xi$. Let h_n and h satisfy $\xi_n = V_{t+1}(\cdot, h_n)$, $\xi = V_{t+1}(\cdot, h)$ and $\lim h_n = h$. By contradiction, suppose that $I_{t,\omega}(\xi_n) \not\to I_{t,\omega}(\xi_n)$. It follows that $V_{t+1}(d^n_{t,\omega}) \not\to V_{t+1}(d_{t,\omega})$. Hence, there exists $\varepsilon \geq 0$ such that for every $N \in \mathbb{N}$ there exists $n \geq N$ such that

$$|V_{t+1}(d_{t,\omega}) - V_{t+1}(d^n_{t,\omega})| \geq \varepsilon > 0.$$

By dynamic consistency it follows that there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $n \geq N$ such that

$$|V_t\left(h_t(\omega), V_{t+1}(d^n_{t,\omega})\right) - V_t\left(h_t(\omega), V_{t+1}(d_{t,\omega})\right)| \geq \epsilon > 0.$$

Observe that by continuity we have $V_t(\omega, h_n) \to V_t(\omega, h)$. Hence, we have arrived at a contradiction. Therefore $I_{t,\omega}(\xi_n) \to I_{t,\omega}(\xi_n)$ as desired.

Now assume that $h_t(\omega) = h'_t(\omega)$ and $I_{t,\omega}(V_{t+1}(\cdot, h)) = I_{t,\omega}(V_{t+1}(\cdot, h'))$. By dynamic consistency, it follows that $h \sim_{t,\omega} h'$. Moreover, if $I_{t,\omega}(V_{t+1}(\cdot, h)) > I_{t,\omega}(V_{t+1}(\cdot, h'))$ then $h_{t,\omega} \succ_{t,\omega} h'$. By Lemma 1 in Gorman (1968) it follows that there exists a continuous function $W_t : X \times V_{t+1} \to \mathbb{R}$ strictly increasing in its second argument such that

$$V_t(\omega, h) = W_t\left(h_t(\omega), I_{t,\omega}\left(V_{t+1}(\cdot, h)\right)\right).$$

Finally observe that by stationarity it holds that $W_t(x, y) = W_{t'}(x, y)$ for every t, t', $x \in X$ and $y \in V_{\max\{t,t'\}+1}$. Therefore, we can set $W \equiv W_0$, which delivers the representation.

I now turn to the necessity of the axioms. It is immediate to check that the recursive representation satisfies axiom 3. To show that the representation satisfies continuity, take $h \in \mathcal{R}$ and a sequence $(f_n)_n$ in $\mathcal{R}$ such that $f_n \succeq_{t,\omega} h$ and $\lim f_n = f$.

This means that $V_t(\omega, f_n) \geq V_t(h)$ for every n so that by sequential continuity of $V_t(\omega, \cdot)$ we obtain that the set

$$\{f \in \mathcal{R} : f \succeq_{t,\omega} h\},$$

is closed. Showing that the set

$$\{f \in \mathcal{R} : h \succeq_{t,\omega} f\},$$

is closed can be done in the same way. Turn now to axiom 2. Let $z \in X$ be such that $u(z) = 0$ and $W(x, u(z)) = u(x)$ (we know z exists by assumption). Now for every $t \leq t', \omega, \omega'$, $d \in \mathcal{D}$, $y, \bar{y}, x, \bar{x} \in X$ it holds that $V_{t'+1}(x_{T-t'}) = V_{t+1}((x_{T-t'}, z_{t-t'}))$. It follows that

$$\begin{aligned}
V_t(\omega, (d_{-t-1}, y, x_{T-t'}, z_{t-t'})) &= W(y, V_{t+1}(x_{T-t'})) \\
\geq V_t\left(\omega, (d_{-t-1}, \bar{y}, \bar{x}_{T-t'}, z_{t-t'})\right) &= W(\bar{y}, V_{t+1}(\bar{x}_{T-t'})) \\
\iff V_{t'}(\omega, (d_{-t'-1}, y, x_{T-t'})) &= W(y, V_{t'+1}(x_{T-t'})) \\
\geq V_{t'}\left(\omega, (d_{-t'-1}, \bar{y}, \bar{x}_{T-t'})\right) &= W(\bar{y}, V_{t'+1}(\bar{x}_{T-t'})),
\end{aligned}$$

which implies that axiom 2 is satisfied. Finally, take $h, h' \in \mathcal{R}$ and (t, ω) with $h_t(\omega) = h'_t(\omega)$. If $h \succeq_{t+1,\omega'} h'$ for every $\omega' \in \mathcal{G}_t(\omega)$ then $V_{t+1}(\omega', h) \succeq_{t+1,\omega'} V_{t+1}(\omega', h')$ which by monotonicity of $I_{t,\omega}$ implies $I_{t,\omega}(V_{t+1}(\cdot, h)) \geq I_{t,\omega}(V_{t+1}(\cdot, h'))$. Since W is strictly increasing in its second variable, it follows that $V_t(\omega, h) \succeq_{t,\omega} V_t(\omega, h')$ as desired. Moreover, if for some $\omega' \in \mathcal{G}_t(\omega)$ the inequality is strict, then by strict monotonicity of $I_{t,\omega}$ we get $I_{t,\omega}(V_{t+1}(\cdot, h)) > I_{t,\omega}(V_{t+1}(\cdot, h'))$ as desired. □

6.3 Proof of Proposition 2

Proof. First observe that if $W_1 = W_2$, then

$$(f_t(\omega), d_{T-t}) \succeq^i_{t,\omega} (f_t(\omega), f_{t+1}, \ldots, f_T) \iff V_{t+1}(d) \geq I^i_{t,\omega}(V^i_{t+1}(\cdot, f)), \tag{7}$$

Now if $\succeq^1_{t,\omega}$ is more risk averse than $\succeq^2_{t,\omega}$ then it is straightforward to check that they rank prospects in $\mathcal{D}$ in the same way. It follows that they must admit recursive representations $(W^i, u^i, (I^i_{t,\omega})_{t,\omega})$, $i = 1, 2$ such that $u^1 = u^2$ and $W^1 = W^2$. By (7) it follows that $I^1_{t,\omega}(\xi) \leq I^2_{t,\omega}(\xi)$ for every $\xi \in \{\xi \in B_0(\mathcal{G}_{t+1}, V_{t+1}) : \xi = V_{t+1}(\cdot, f), f \in \mathcal{R}\}$. The converse follows immediately by (7). □

6.4 Proof of Theorem 2

Proof of Proposition 2. First observe that (5) is equivalent to axiom 6. The remainder of the proof uses arguments from Selden and Stux (1978) (proof of Lemma 1). I prove sufficiency of the axioms in the case of $\succeq_0$, and using consequentialism the result follows for $\succeq_{t,\omega}$ analogously. I claim that for every $h \in \mathcal{F} \setminus \mathcal{R}$, there exists $\bar{c} = (h_0, c_1, \dots, c_T) \in \mathcal{D}$ such that $\bar{c} \sim_0 h$ and

$$c_t = \left[\mathbb{E}_{\prod_{\tau=1}^{t} P(s_\tau)} h_t^\alpha\right]^{\frac{1}{\alpha}},$$

which establishes the representation since $\succeq_0$ has an EZ representation so that

$$V_0((h_0, c_1, \dots, c_T))) = u(h_0) + \sum_{t=1}^{T} \beta^j u(c_j),$$

where $u(x) = \frac{x^\rho}{\rho}$, as desired. First, for every $s^{t-1} = (s_1, \dots, s_{t-1})$, let

$$c_t^1(s^{t-1}) = \left[\mathbb{E}_{P(s_t)} h_t^\alpha\right]^{\frac{1}{\alpha}}.$$

Observe that axioms 7 and 8, we have

$$h \sim_0 (h_0, \dots, h_{T-1}, c_T^1).$$

Now by further applying axioms 7 and 8 we get

$$(h_0, \dots, h_{T-1}, c_T^1) \sim_0 (h_0, \dots, c_{T-1}^1, c_T^1).$$

By axiom 5,

$$(h_0, \dots, c_{T-1}^1, c_T^1) \sim_0 (h_0, \dots, c_{T-1}^1, \hat{c}_T^1),$$

where $\hat{c}_T^1(s_1, s_{T-2} \dots, \cdot, s_T)$ is constant and

$$\hat{c}_T^1(s_1, \dots, \cdot) = c^1(s_1, \dots, \cdot).$$

By another application of axioms 7 and 8 we obtain:

$$(h_0, \dots, c_{T-1}^1, \hat{c}_T^1) \sim_0 (h_0, \dots, c_{T-1}^1, c_T^2),$$

where

$$c_T^2 = \left[\mathbb{E}_{P(s_{T-1})P(s_T)} h_T^\alpha\right]^{\frac{1}{\alpha}}.$$

Proceeding as in the previous to steps, we obtain at step t

$$h \sim_0 (h_0, \dots, c^1_{T-t+1}, \dots, c^{t-1}_{T-1}, c^t_T),$$

where

$$c^t_j = \left[\mathbb{E}_{\prod_{\tau=T-t+1}^{j} P(s_\tau)} h^\alpha_j\right]^{\frac{1}{\alpha}}.$$

Specifically, after T steps we get

$$h \sim_0 (h_0, c_1, \dots, c_T),$$

as desired.

I turn to the necessity of the axioms. It is immediately verified that all axioms are satisfied, except axiom 5. I prove that the representation satisfies indifference to timing. Take h, h' such that for some $\bar{s}^t = (\bar{s}_1, \dots, \bar{s}_t)$ with $1 \leq t \leq T-2$ the act

$$(h_0, h_1(\bar{s}_1), h_2(\bar{s}_1, \bar{s}_2), \dots, h_t(\bar{s}^t), \dots, h_T(\bar{s}^t, \cdot)),$$

resolves earlier than

$$(h'_0, h'_1(\bar{s}_1), h'_2(\bar{s}_1, \bar{s}_2), \dots, h'_t(\bar{s}^t), \dots, h'_T(\bar{s}^t, \cdot)),$$

and $h_\tau(s_1, \dots, s_\tau) = h'_\tau(s_1, \dots s_\tau)$ for every $(s_1, \dots, s_\tau)$ with $\tau < t$ and $h_\tau(s_1, \dots, s_\tau) = h'_\tau(s_1, \dots s_\tau)$ for every $(s_1, \dots s_\tau)$ such that $\tau \geq t$ and $(s_1, \dots, s_t) \neq (\bar{s}_1, \dots, \bar{s}_t)$. Then we have for $t' \leq t$

$$V_{t'}((s_1, \dots, s_{t'}), h) - V_{t'}((s_1, \dots, s_{t'}), h') \propto$$
$$\sum_{j=0}^{T-t} \beta^j u\left(\left[\mathbb{E}_{\prod_{\tau=1}^{j} P(s_{t+\tau})} h^\alpha_{t+j}\right]^{\frac{1}{\alpha}} \right) - \sum_{j=0}^{T-t} \beta^j u\left(\left[\mathbb{E}_{\prod_{\tau=1}^{j} P(s_{t+\tau})} h'^\alpha_{t+j}\right]^{\frac{1}{\alpha}} \right).$$

Observe that by assumption on h, h' we have

$$\mathbb{E}_{\prod_{\tau=1}^{j} P(s_{t+\tau})} h^\alpha_{t+j} = \mathbb{E}_{\prod_{\tau=1}^{j} P(s_{t+\tau})} h'^\alpha_{t+j},$$

for $j = 0, \dots, T-t$. Therefore $V_{t'}((s_1, \dots, s_{t'}), h) - V_{t'}((s_1, \dots, s_{t'}), h') = 0$ whence the result follows.

□

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