Arbitrage Pricing in Convex, Cash-Additive Markets*

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Abstract

We consider superhedging and no-arbitrage pricing in markets with a convex and cash-additive structure and derive an explicit functional form for the super-replication price. Using convex duality methods, we show that the superhedging price maximizes the difference between the expected payoff and a confidence function that accounts for the reliability of the probability used in pricing. We demonstrate that the existence of a strictly positive probability within the domain of the confidence function, which maximizes the super-replication price for a specific payoff and acts as a lower bound for all other payoffs, is necessary and sufficient to prevent arbitrage opportunities. Furthermore, we explore entropy pricing as a notable example of a super-replication pricing functional and provide conditions on the market structure under which the super-replication price takes the form of entropy pricing. We show that the confidence function in entropy pricing can be expressed using the Kullback-Leibler divergence.

1 Introduction

In financial markets, the order size can affect a security's price per unit, as many empirical studies show (Almgren, Thum, Hauptmann and Li 2005, Moro, Vicente, Moyano, Gerig,

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Farmer, Vaglica, Lillo and Mantegna 2009, Tóth, Lemperiere, Deremble, Lataillade, Kockelkoren and Bouchaud 2011, Bacry, Iuga, Lasnier and Lehalle 2015, and Donier and Bonart 2015). This literature consistently demonstrate that the temporary price impact increases approximately with the square root of the order size.

Lécuyer and Martins-da Rocha (2021) extend the fundamental theorem of asset pricing in convex market structures to account for this phenomenon, showing that the price of a security is convex in the quantity traded. Here, we add two natural assumptions to the model studied by Lécuyer and Martins-da Rocha (2021). We assume the availability of a riskless security and that the pricing functional is cash-additive. Cash-additivity is a well-known postulate in Mathematical Finance that models the absence of frictions on the market for riskless securities (Föllmer, Schied and Lyons 2004).

Our first main result, Theorem 3.2, shows that the resulting super-replication price has an explicit functional form that depends on a confidence function α. It captures the highest deviation from an idealized frictionless market, reflecting the unfavorable impact of market friction on the pricing of a financial instrument. We also show that the super-replication price is well-defined and continuous.

Our second main result, Theorem 3.3, shows that it is necessary and sufficient for the super-replication price to be supported by a frictionless no-arbitrage price to satisfy the no-arbitrage conditions. When the super-replication price is linear, this results in the standard characterization of no-arbitrage.

In Section 4, we explore an important example of super-replication pricing, entropy pricing, and provide conditions for the market structure such that the super-replication price has an entropy pricing form. The main condition is segmented-additivity, a condition that states that the value of a combined portfolio equals the sum of the values of its individual components when those components yield non-zero payoffs in distinct, non-overlapping states of the world. In this case, we show that the confidence function α is given by the Kullback-Leibler divergence. This example provides a major tractable example of a super-replication price.

Overall, our results extend the fundamental theorem of asset pricing in convex market structures, provide a framework to account for market impact, transaction costs, and taxes, and offer insights into the super-replication price and its relation to the traditional concept of no-arbitrage.

This article contributes to the growing body of literature that extends the fundamental theorem of asset pricing (FTAP) to account for frictions present in financial markets such as transaction costs, taxes, and market impact. The FTAP was first demonstrated in frictionless markets by Harrison and Kreps (1979). Subsequently, Jouini and Kallal (1995) and Luttmer (1996) extended the FTAP to sublinear pricing rules. When markets are complete, Cerreia-Vioglio, Maccheroni and Marinacci (2015) characterized a pricing rule satisfying cash-additivity, monotonicity, and put-call parity. Burzoni, Riedel and Soner (2021) extended the FTAP further by introducing a more general setting that encompasses market uncertainty. Under the assumption that the set of net trades is a convex cone, they show that markets viability is equivalent to the existence of a pricing rule taking the form of a lower semi-continuous sublinear martingale expectation with full support.

The pricing rules considered by Jouini and Kallal (1995), Luttmer (1996), Cerreia-Vioglio, Maccheroni and Marinacci (2015), and Burzoni, Riedel and Soner (2021) are positively homogeneous, meaning that the size of an order executed in the markets does not modify the unitary price of the order, i.e., unitary prices are constant in quantity traded. As a result, these pricing rules cannot account for market impact, the fact that unitary prices actually increase with the quantity traded.

Several other models exist that can account for market impact assuming the pricing rule is convex. For example, Jouini and Kallal (1999) showed that a convex pricing rule is viable if, and only if, there is no asymptotic free-lunch. Lécuyer and Martins-da Rocha (2021) proposed a new approach making assumptions on the primitives of the model, the market structure, which is the actual price paid for a portfolio, and the payoffs received. They assumed the market structure is convex and determined the relevant concept of no-arbitrage, which they called robust no-arbitrage. They also derived the super-replication pricing rule from the primitives and show that it is convex. Moreover, it is viable (that is it satisfies robust no arbitrage) if the market structure satisfies no robust arbitrage. However, due to the generality of their model, they do not provide an explicit expression for the super-replication pricing rule making it challenging to use in practice. Therefore, in this article we address a special case of this model by assuming that a riskless asset is available and that the market structure is cash-additive in addition to being convex.

Section 2.1 introduces the notation and the main mathematical objects used in the paper. Section 2.2 presents the model and 3 contains the main results of the paper. Section 4 considers the special case of entropic pricing rules. The proofs are all in the Appendix.

2 Convex Cash-Additive Markets

2.1 Preliminaries

We use the following notations and definitions. Given a non-empty finite set K, we call an element $X \in \mathbb{R}^K$ a vector and we denote it by $X = (X(k))_{k \in K}$ or $X = (X_k)_{k \in K}$. For every $X \in \mathbb{R}^K$, we denote $\operatorname{supp} X := \{k \in K : X(k) \neq 0\}$ the support of X, i.e. the set of points at which X is non-zero. A vector $X \in \mathbb{R}^K$ is nonnegative (strictly positive) when $X(k) \geqslant 0$ (resp. $X(k) > 0$) for all $k \in K$. The set of nonnegative (strictly positive) vectors is denoted by $\mathbb{R}^K_+$ (resp. $\mathbb{R}^K_{++}$). Given $X \in \mathbb{R}^K$, $\|X\|$ denotes the standard Euclidean norm of X. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is spaces), $U \subseteq X$ is open, and $F : U \to Y$. The Gateaux derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at $X \in \mathbb{R}^n$ in the direction $Y \in \mathbb{R}^m$ is defined as

$$f'(X;Y) = \lim_{h \to 0} \frac{f(X + hY) - F(X)}{h}.$$

We say that f is Gateaux differentiable if $f'(X;Y)$ exists finite for every X and Y. A function $f : \mathbb{R}^K \to \mathbb{R}$ is sub-additive when $f(X+Y) \leqslant f(X) + f(Y)$ for every $X, Y \in \mathbb{R}^K$, and positively homogeneous when $f(\lambda X) = \lambda f(X)$ for every $\lambda > 0$ and $X \in \mathbb{R}^K$. Sub-additive and positively homogeneous functions are called sublinear.

We let $\Delta(K)$ denote the set of probability measures defined on the probability space $(K, 2^K)$. A probability $P \in \Delta(K)$ has full support if $P(k) > 0$ for every $k \in K$. Considering two probabilities $P, Q \in \Delta(K)$, the notation $P \ll Q$ means that P is absolutely continuous with respect to Q in the sense that for every $A \subseteq K$, we have $Q(A) = 0 \implies P(A) = 0$. The expectation under $P \in \Delta(K)$ of a random vector $X \in \mathbb{R}^K$ is denoted by $\mathbb{E}_P[X] := P \cdot X$.

2.2 The Market

We consider two points in time: $t = 0$ and $t = 1$. The second point in time, $t = 1$, is characterized by uncertainty, represented by a finite set Ω of possible states of nature. At the point in time 0, agents can access financial markets, where a set of J securities is available for trading. One of these securities is a riskless asset that pays 1 in every state of nature.

Agents can form portfolios of securities, denoted by $\theta \in \mathbb{R}^J$. The first coordinate of the portfolio, θ_1, represents the quantity of the riskless asset that is purchased (positive) or sold (negative), while the remaining coordinates, θ_j for $j \in J \setminus \{1\}$, represent the quantities of the other securities purchased or sold.

We assume that portfolios are not subject to any restrictions, and their prices are represented by a function $p : \mathbb{R}^J \to \mathbb{R}$. Here, $p(\theta)$ is the cost of trading portfolio θ at $t = 0$. A mapping $G : \mathbb{R}^J \to \mathbb{R}^\Omega$ models the payoffs of portfolios, $G(\theta, \omega) \in \mathbb{R}$ denoting the payoff of portfolio θ in state ω. The couple (p, G) is called a market structure.

Definition 2.1. The market structure (p, G) is

1. **convex** if p is a convex function satisfying $p(0) = 0$ and $G(\cdot, \omega)$ is concave for every $\omega \in \Omega$, and $G(0) = 0$,

2. **cash-additive** if it satisfies the following properties for all $\theta \in \mathbb{R}^J$ and all $k \in \mathbb{R}$:

$$p(\theta + ke_1) = p(\theta) + k \quad \text{and} \quad G(\theta + ke_1) = G(\theta) + k\mathbf{1}_\Omega,$$

   where $\mathbf{1}_\Omega \in \mathbb{R}_\Omega$ is the vector with all coordinates equal to 1 and $e_1 \in \mathbb{R}^J$ is the vector whose first coordinate is equal to 1 and all other coordinates are equal to 0.

Remark 2.1. We assume zero interest rates. Positive interests can easily be accommodated.

In Lécuyer and Martins-da Rocha (2021), the authors extend the fundamental theorem of asset pricing for convex market structures. In this paper, we focus on the special case where the market structure is *cash-additive* in addition to being convex. In the rest of the paper, we thus maintain the following assumption.

Assumption 2.1. The market structure (p, G) is convex and cash-additive.

Cash-additivity has the important consequence that every payoff can be superhedged because a trivial superhedge is given by a sufficiently large quantity of the riskless security.

Before we come to the fundamental theorem, let us discuss some archetypical markets that are covered by our setup. The frictionless case, when both p and G are linear, is naturally a particular case of our setup.

Example 2.1 (Transaction Costs)**.** Consider the situation when the assets are traded at bid-ask prices $0 \leq q_j^B \leq q_j^A$ and payoffs x_j are linear for every asset j as discussed in Araujo, Chateauneuf and Faro (2018) and the appendix of Beissner and Riedel (2019). The riskless asset is frictionless with $q_1^B = q_1^A = x_1 = 1$. Write $\theta_j^+ = \max\{\theta_j, 0\}$ and $\theta_j^- = \max\{-\theta_j, 0\}$ for the long and short positions, respectively. The price of a portfolio at time 0 is given by the sublinear functional

$$p(\theta) = \sum_{j=1}^{J} \left(\theta_j^+ q_j^A - \theta_j^- q_j^B \right)$$

and the payoff is given by the linear functional $G(\theta) = \sum_{j=1}^{J} \theta_j x_j$. △

Example 2.2 (Incomplete and Ambiguous Markets)**.** Ambiguous pricing rules have recently been studied by Beissner and Riedel (2019) who develop the general theory of equilibrium for such markets and Araujo, Chateauneuf and Faro (2012) discuss the structure of frictionless parts of the market. Both papers immediately start with the (sublinear) superhedging functional that we discuss below. △

Example 2.3 (Entropy Pricing)**.** Consider the situation when there are three states of the nature, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, three assets ($J = 3$) traded on the markets and the price of a portfolio at time 0 is given by the functional

$$p(\theta) = \log\Big(\frac{1}{3}e^{\theta_1+\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+2\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+\theta_2+2\theta_3}\Big),$$

for all $\theta \in \mathbb{R}^J$, while the payoffs of portfolios are linear and given by the mapping $G(\theta) = (\theta_1 + \theta_2 + \theta_3, \theta_1 + 2\theta_2 + \theta_3, \theta_1 + \theta_2 + 2\theta_3)$. In this case, the market structure (p, G) describes frictions only related to pricing. Not trading at time 0 is not costly, i.e. $p(0) = \log(1) = 0$. Moreover, we have that

$$\begin{aligned} p(\theta + ke_1) =& \log\Big(\frac{1}{3}e^{1+\theta_1+\theta_2+\theta_3} + \frac{1}{3}e^{1+\theta_1+2\theta_2+\theta_3} + \frac{1}{3}e^{1+\theta_1+\theta_2+2\theta_3}\Big) \\ =& \log\Big(\Big(\frac{1}{3}e^{\theta_1+\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+2\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+\theta_2+2\theta_3}\Big)e^k\Big) \\ =& \log\Big(\frac{1}{3}e^{\theta_1+\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+2\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+\theta_2+2\theta_3}\Big) + k \\ =& p(\theta) + k. \end{aligned}$$

Finally, observe that p is the composition of the convex function $X \mapsto \log\Big(\sum_{i=1}^3 \frac{1}{3}e^{X(\omega_i)}\Big)$ with the linear function $G(\theta) = (\theta_1 + \theta_2 + \theta_3, \theta_1 + 2\theta_2 + \theta_3, \theta_1 + \theta_2 + 2\theta_3)$, so that it is convex as well. Hence, we can conclude that (p, G) is convex and cash-additive. △

3 Arbitrage and the Fundamental Theorem

In this section, we give a definition of arbitrage for our market structures and provide a version of the fundamental theorem of asset pricing. We adopt the following strengthening standard notion of absence of arbitrage (e.g., see Ross (2005)).

Definition 3.1 (Robust no-arbitrage Lécuyer and Martins-da Rocha (2021))**.** The convex market structure (p, G) satisfies the robust no-arbitrage property if p and G are Gateaux differentiable and there exists a portfolio $\theta^0 \in \mathbb{R}^J$ such that for any direction $\eta \in \mathbb{R}^J$, the conditions

$$G'\Big(\theta^0; \eta\Big) \geqslant 0 \quad \text{and} \quad p'\Big(\theta^0; \eta\Big) \leqslant 0,$$

imply

$$G'\Big(\theta^0; \eta\Big) = 0 \quad \text{and} \quad p'\Big(\theta^0; \eta\Big) = 0.$$

As discussed by Lécuyer and Martins-da Rocha (2021), this notion of no-arbitrage is a stronger requirement than the standard no-arbitrage condition, but they coincide when p and G are linear. Indeed, observe that when p and G are both linear, we have that $p'(\theta^0; \eta) = p(\eta)$ and $G'(\theta^0; \eta) = G(\eta)$, so that the robust no-arbitrage condition becomes equivalent to

$$[G(\eta) \geqslant 0 \text{ and } p(\eta) \leqslant 0] \Longrightarrow [G(\eta) = 0 \quad \text{and} \quad p(\eta) = 0],$$

which is exactly the standard notion of the absence of arbitrage.

We introduce a new characterization of this robust no-arbitrage specific to the cash-invariant case.

Theorem 3.1 (FTAP for convex and cash additive market structures)**.** *Consider a convex market structure (p, G) such that both p and G are Gateaux differentiable. Then (p, G) satisfies the robust no-arbitrage property if and only if there exists a measure $\mu \in \Delta(\Omega)$ with full support such that for all portfolios $\theta \in \mathbb{R}^J$*

$$\mathbb{E}_\mu G(\theta) \leq p(\theta).$$

Notice that our characterization is stronger than that in Lécuyer and Martins-da Rocha (2021) obtained for general convex market. Indeed, here μ is a probability measure—what is commonly referred to as a *martingale* probability measure, while in their case it is only restricted to be a strictly positive vector. Hence, convex cash additive market structures have a majpr advantage compared to arbitrary convex market structure. We illustrate this result in the following example.

Example 3.1 (Example 2.3 continued)**.** In this case we have that the price function is

$$p(\theta) = \log\Big(\frac{1}{3}e^{\theta_1+\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+2\theta_2+\theta_3} + \frac{1}{3}e^{\theta_1+\theta_2+2\theta_3}\Big),$$

and a payoff function

$$G(\theta) = (\theta_1 + \theta_2 + \theta_3, \theta_1 + 2\theta_2 + \theta_3, \theta_1 + \theta_2 + 2\theta_3).$$

Because $x \mapsto e^x$ is convex, if we let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ then it holds that for every $\theta \in \mathbb{R}^J$

$$p(\theta) \geqslant \log\left(e^{\theta_1 + \frac{2\theta_2 + 2\theta_3}{3}}\right) \geqslant \theta_1 + \frac{2\theta_2 + 2\theta_3}{3} = \mathbb{E}_\mu G(\theta).$$

Hence, μ is a martingale measure with full support. By Theorem 3.1, since both p and G are Gateaux differentiable, it follows that (p, G) satisfies the robust no-arbitrage property. $\triangle$

3.1 Super-replication Prices and no-arbitrage

Rather than analyzing an agent's optimal portfolio, we can focus directly on the cost at time $t = 0$ required to implement a specific consumption plan at $t = 1$. More precisely, we ask: what is the amount of initial resources the agent should allocate at $t = 0$ in order to achieve a specified random consumption X at $t = 1$ through trading portfolios? The natural approach is to determine the least expensive portfolio θ such that $G(\theta) \geqslant X$.

Definition 3.2. The **super-replication price** associated with the market structure (p, G) is the function $\pi^p : \mathbb{R}^\Omega \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\forall X \in \mathbb{R}^\Omega, \quad \pi^p(X) := \inf\{p(\theta) \ : \ \theta \in \mathbb{R}^J \quad \text{and} \quad X \leqslant G(\theta)\}.$$

In this section, we study the super-replication functional in general and provide a dual representation. Cash-additivity and convexity transfer from the market structure to the super-replication price. Moreover, with finitely many values, every security is bounded, and can thus be superhedged with cash. Due to cash-additivity, the super-replication price is thus bounded above.

Proposition 3.1. *If the market structure (p, G) is convex and cash additive, then the super-replication price π^p convex, monotone, continuous, and cash-additive in the sense that, for every $X \in \mathbb{R}^\Omega$ and $k \in \mathbb{R}$, we have*

$$\pi^p(X + k\mathbf{1}_\Omega) = \pi^p(X) + k.$$

Moreover, is finite, i.e. $\pi^p(X) < \infty$.

The properties of convexity, cash-additivity, and continuity of the super-replication price enable us to derive a dual representation. Up to a sign convention, our super-replication price is a convex risk measure and thus, it can be expressed as the maximum of expected payoffs minus a confidence function over the set of probability measures, cf. Föllmer and Schied (2016).

Theorem 3.2. *For all payoffs $X \in \mathbb{R}^\Omega$, the super-replication price π^p satisfies*

$$\pi^p(X) = \max_{P \in \Delta(\Omega)} (\mathbb{E}_P(X) - \alpha(P)), \tag{1}$$

for a confidence function $\alpha : \Delta(\Omega) \to \mathbb{R}_+ \cup \{+\infty\}$ given by

$$\alpha(P) = \sup_{X \in \mathbb{R}^\Omega} (\mathbb{E}_P(X) - \pi^p(X)).$$

It is well known that we can represent the super-replication price with respect to a minimal confidence function. Let $\mathcal{A}_{\pi^p}$ be the set

$$\mathcal{A}_{\pi^p} = \{X \in \mathbb{R}^\Omega \text{ s.t. } \pi(X) \leqslant 0\}.$$

Proposition 3.2. *The confidence function $\alpha_{\min} : \Delta(\Omega) \to [0, \infty)$ given by*

$$\alpha_{\min}(P) = \sup_{X \in \mathcal{A}_{\pi^p}} \mathbb{E}_P(X),$$

is the minimal confidence function representing the super-hedging price π^p, that is, every confidence function α satisfying Equation 1 is such that $\alpha(P) \geqslant \alpha_{\min}(P)$ for all $P \in \Delta(\Omega)$.

We take up our examples from above.

Example 3.2 (Example 2.1 continued)**.** In the context of transaction costs, Araujo, Chateauneuf and Faro (2018) show that the superhedging functional can be written as the supremum of expected values over a polytope

$$\mathcal{Q} = \{Q_1, \dots, Q_n\}$$

of extremal martingale measures Q_k that satisfy

$$q_j^B \leq E^{Q_k} x_j \leq q_j^A$$

for $j = 1, \dots, J$. We thus have

$$\pi^p(X) = \max_{Q \in \mathcal{Q}} \mathbb{E}_Q(X).$$

△

Example 3.3 (Example 2.3 continued)**.** In Theorem 4.1 below, we show that given (p, G) from Example 2.3, π is such that the confidence function α is given by the Kullback-Leibler divergence, i.e. for $Q \in \Delta(\Omega)$ given by $Q(\omega_1) = Q(\omega_2) = Q(\omega_3) = \frac{1}{3}$ such that

$$\alpha(P) := R(P\|Q) = \sum_{\omega \in \Omega} P(\omega) \log\left(\frac{P(\omega)}{Q(\omega)}\right),$$

whenever $P \ll Q$, and $R(P\|Q) = +\infty$ otherwise. In this case, the further a probability is from Q as measured by relative entropy, the lower its "weight". In words, this confidence function describes an investor who, in the pricing of payoffs, is most confident about $Q \in \Delta(\Omega)$ and is less confident about probabilities that are farther away from Q as measured by the Kullback-Leibler divergence. $\triangle$

The previous example shows that cash-additivity does not result in positive homogeneity unlike in Example 3.2.

We now provide a version of the fundamental theorem of asset pricing for the super-replication price.

Definition 3.3. The market structure (p, G) does not admit arbitrage if there exists a payoff $X_0 \in \mathbb{R}^\Omega$ such that for all payoffs $X \in \mathbb{R}^\Omega$, we have

$$[X \geqslant 0 \quad \text{and} \quad \pi^p(X + X_0) - \pi^p(X_0) \leqslant 0] \implies X = 0.$$

We introduce a new characterization of no-arbitrage specific to the cash-invariant super-replication price case. We say that a super-replication price is supported by a no-arbitrage frictionless price if there exists a probability with full support in the domain of the confidence function such that for every payoff, the expected value of the payoff with respect to this probability is a lower bound of the super-replication price of the payoff.

Definition 3.4. We say that π^p is supported by a no-arbitrage frictionless price if there exist a payoff $X_0 \in \mathbb{R}^\Omega$ and a probability $\mu \in \Delta(\Omega)$ such that

$$\alpha(\mu) = 0, \quad \pi^p(X_0) = \mathbb{E}_\mu(X_0),$$

and

$$\pi^p(X) \geqslant \mathbb{E}_\mu(X) \quad \text{for all} \quad X \in \mathbb{R}^\Omega.$$

Theorem 3.3 (FTAP for the super-replication price)**.** *The convex market structure (p, G) does not admit arbitrage if and only there exists a martingale measure with full support.*

In particular, if the super-replication price π^p is consistent with the absence of arbitrage, then the price functional p is also supported by the same no-arbitrage frictionless price in the sense that for all $\theta \in \mathbb{R}^J$ such that $G(\theta) \geqslant X$, we have

$$p(\theta) \geqslant \mathbb{E}_\mu(X),$$

with μ the probability with full support such that $\pi^p(X_0) = \mathbb{E}_\mu(X_0)$ for some $X_0 \in \mathbb{R}^J$ and $\pi^p(X) \geqslant \mathbb{E}_\mu(X)$ for all $X \in \mathbb{R}^\Omega$. We illustrate this result in the following two examples.

Example 3.4 (Example 3.2 continued)**.** Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the confidence function

$$\alpha(P) = \begin{cases} 0 & \text{for } P \in C \\ +\infty & P \notin C, \end{cases}$$

where

$$C = \{P \in \Delta(\Omega) : P(\omega_1) = \frac{1}{3}, P(\omega_2) + P(\omega_3) = \frac{2}{3}\}.$$

Let $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $X_0 = (0, 1, 1)$. Then

$$\pi^p(X_0) = \max_{P \in C} \mathbb{E}_P X_0 = \mathbb{E}_\mu X_0 = \frac{2}{3}.$$

Moreover, since

$$\pi^p(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X,$$

for every $X \in \mathbb{R}^\Omega$ it holds that

$$\pi^p(X) \geqslant \mathbb{E}_\mu X.$$

Hence by Theorem 3.1 we can conclude that π^p is a no-arbitrage price. △

Example 3.5 (Example 2.3 continued)**.** Suppose again $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Recall that in this example the confidence function is given by the Kullback-Leibler divergence,

$$\alpha(P) = R(P\|Q) = \sum_{\omega \in \Omega} P(\omega) \log\left(\frac{P(\omega)}{Q(\omega)}\right),$$

whenever $P \ll Q$, and $R(P\|Q) = +\infty$ otherwise, where $\mu = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Let $X_0 = (1, 1, 1)$ and $Q = \mu$. Then $\pi^p(X_0) = \mathbb{E}_\mu(X_0) = 1$ and moreover

$$\pi^p(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X - R(P\|Q) \geqslant \mathbb{E}_\mu X - R(\mu\|\mu) = \mathbb{E}_\mu X,$$

for every $X \in \mathbb{R}^\Omega$. Hence by Theorem 3.1 we can conclude that π^p is a no-arbitrage price.

More in general, whenever α is given the Kullback-Leibler divergence with respect to $Q \in \Delta(\Omega)$ with full support, the martingale probability μ is given by Q. △

4 Entropic super-replication price

Entropy pricing is very common in finance (e.g., see Buchen and Kelly (1996), Leitner (2008) or the review in Zhou, Cai and Tong (2013) for a discussion of its relevance).

Definition 4.1. Given a probability $Q \in \Delta(\Omega)$, the entropic super-replication price is given by

$$\pi(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X - R(P||Q),$$

where $R(\cdot\|\cdot)$ is the Kullback-Leibler divergence, i.e.

$$R(P\|Q) = \sum_{\omega \in \Omega} P(\omega) \log\left(\frac{P(\omega)}{Q(\omega)}\right),$$

whenever $P \ll Q$, and $R(P\|Q) = +\infty$ otherwise.

Recall that due to a key duality result (e.g., see Dupuis and Ellis 2011), we have that

$$\pi(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X - R(P||Q) = \log(\mathbb{E}_Q(e^X)). \tag{2}$$

Observe that because of this duality result, it is immediate to check that π satisfies cash-additivity. We provide a foundation for such π based on more primitive conditions on the market structure (p, G). Here we think of p as assigning a price to each portfolio measured in a *logarithmic scale*. Indeed, with relative entropy pricing prices are measured in a logarithmic scale. Therefore, e^p quantifies prices measured in the *arithmetic scale*.

Definition 4.2 (Segmented addivitity)**.** The price functional p satisfies additivity for segmented portfolios if for every $\theta, \theta' \in \mathbb{R}^J$

$$\operatorname{supp} G(\theta) \cap \operatorname{supp} G(\theta') = \emptyset \implies e^{p(\theta+\theta')} = e^{p(\theta)} + e^{p(\theta')} - 1.$$

Such a condition requires additivity of the price function (when prices are measured in the arithmetic scale) to hold when two portfolios are "segmented" in an intuitive way: we say that they are segmented when they have non-zero payoffs in different, non-overlapping states of the world.

Say that the payoff mapping G is monotone if $G(\theta) \geqslant G(\theta')$ whenever $\theta \geqslant \theta'$. The price functional $p : \mathbb{R}^J \to \mathbb{R}$ is said to be strictly monotone if for every $\theta, \theta' \in \mathbb{R}^J$

$$G(\theta) \geqslant G(\theta') \implies p(\theta) \geqslant p(\theta'),$$

and $p(\theta) > p(\theta')$ if it further holds that $G(\theta)(\omega) > G(\theta')(\omega)$ for some $\omega \in \Omega$.

The next example illustrates segmented additivity.

Example 4.1 (Example 2.3 continued)**.** Let $\theta = (-1, 1, 0)$ and $\theta' = (-1, 0, 1)$. Then $G(\theta) = (0, 1, 0)$ and $G(\theta') = (0, 0, 1)$. Therefore it holds that $\operatorname{supp} G(\theta) \cap \operatorname{supp} G(\theta') = \emptyset$.

Now observe that

$$e^{p(\theta+\theta')} = \frac{2}{3}e + \frac{1}{3} = \frac{1}{3}e + \frac{1}{3}e + \frac{4}{3} - 1 = e^{p(\theta)} + e^{p(\theta')} - 1.$$

In words, these two portfolios never pay a non-zero amount in the same state of the world. Therefore, in this case the price of the portfolio $\theta + \theta'$ can be additively determined through the formula in Definition 4.2. Moreover, in this case it is immediate to check that $\pi(X) = \log\Big(\sum_{i=1}^{3} \frac{1}{3} e^{X(\omega_i)}\Big)$. Hence in this case we obtain:

$$\pi(X) = \log\Big(\mathbb{E}_Q\Big(e^X\Big)\Big) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X - R(P||Q),$$

where $Q(\omega_1) = Q(\omega_2) = Q(\omega_3) = \frac{1}{3}$. △

The following result generalizes the previous example.

Theorem 4.1. *Suppose that $|\Omega| \geqslant 3$. Assume that (p, G) is complete with G monotone and linear. Further, assume that p is strictly monotone and satisfies segmented additivity. Then there exists $Q \in \Delta(\Omega)$ with full support such that*

$$\pi(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X - R(P||Q) = \log\Big(\mathbb{E}_Q\Big(e^X\Big)\Big),$$

for every $X \in \mathbb{R}^\Omega$.[1]

Proof. See the Appendix. □

It is important to note that this pricing functional also satisfies the additivity property

$$\pi(X + Y) = \pi(X) + \pi(Y),$$

whenever X and Y are independent in the probability space $(\Omega, 2^\Omega, Q)$. Such an additivity property is particularly significant in the context of portfolio pricing, as it implies that portfolios consisting of unrelated risks should be priced independently of one another.

Furthermore, entropy pricing exhibits strict convexity, leading to the inequality $\pi(\alpha X) < \alpha\pi(X)$ for every $\alpha \in (0, 1)$. This result can be attributed to the presence of market frictions. Notably, Theorem 4.1 demonstrates that these frictions are only related to the price function p and not to the payoff mapping G. Therefore, this result suggests that entropic pricing is not suitable when frictions arise, for example, from taxes.

[1]The more general case in which $\alpha(P) = aR(P||Q)$ for some $a > 0$ can be dealt with by imposing the additivity condition $e^{ap(\theta+\theta')} = e^{ap(\theta)} + e^{ap(\theta')} - 1$.

5 Conclusion

This paper discusses hedging and the absence of arbitrage in convex cash-additive markets. We provide a version of the fundamental theorem of asset pricing in this setting. By focusing on the super-replication prices, we introduce the concept of a confidence function that reflects market frictions, such as transaction costs and taxes. As a major special case, we characterize the case of entropy pricing super-replication functionals. Hence, our findings extend traditional asset pricing frameworks to better accommodate real-world market imperfections.

6 Appendix

6.1 Proofs

Proof of Theorem 3.1. By Theorem 3.3 in Lécuyer and Martins-da Rocha (2021) it follows that (p, G) satisfies robust no-arbitrage if and only if there exists a strictly positive vector $\mu \in \mathbb{R}^{\Omega}_{++}$ and a portfolio $\theta^\star \in \mathbb{R}^J$ such that

$$p(\theta) - p(\theta^\star) \geqslant \mu \cdot [G(\theta) - G(\theta^\star)], \tag{3}$$

for all $\theta \in \mathbb{R}^J$.

Hence, if we assume that (p, G) satisfies the no-arbitrage condition, then by (3) it follows that if we set $\theta = \theta^\star - 1$, then

$$1 \geqslant \sum_{\omega \in \Omega} \mu(\omega).$$

If we substitute into (3) $\theta = \theta^\star + 1$, then by the same reasoning we obtain that

$$1 \leqslant \sum_{\omega \in \Omega} \mu(\omega).$$

We can therefore conclude that $\mu \in \Delta(\Omega)$.

Conversely, if there exists a martingale measure μ with full support such that for all portfolios $\theta \in \mathbb{R}^J$

$$\mathbb{E}_\mu G(\theta) \leq p(\theta),$$

then (3) holds with $\theta^\star = 0$, delivering the desired result. □

Proof of Proposition 3.1. Monotonicity of the super-replication price was showed in Lécuyer and Martins-da Rocha (2021). First, we are going to show that π^p is cash-additive when (p, G) is cash-additive. Let $X \in \mathbb{R}^\Omega$ and $k \in \mathbb{R}$. Then by definition,

$$\pi^p(X + k\mathbf{1}_\Omega) = \inf\{p(\eta) : G(\eta) \geqslant X + k\mathbf{1}_\Omega\}.$$

This is equivalent to

$$\pi^p(X + k\mathbf{1}_\Omega) = \inf\{p(\theta) : G(\theta) - k\mathbf{1}_\Omega \geqslant X\}.$$

By cash-additivity of G, we have

$$\pi^p(X + k\mathbf{1}_\Omega) = \inf\{p(\theta) : G(\theta - ke_1) \geqslant X\}.$$

We pose $\eta_1 = \theta_1 - k$ and $\eta_j = \theta_j$ for all $j \in J \setminus \{1\}$, then

$$\pi^p(X + k\mathbf{1}_\Omega) = \inf\{p(\eta + ke_1) : G(\eta) \geqslant X\}.$$

By cash-additivity of p, it is equivalent to

$$\pi^p(X + k\mathbf{1}_\Omega) = \inf\{p(\eta) + k : G(\eta) \geqslant X\}.$$

Hence the desired result

$$\pi^p(X + k\mathbf{1}_\Omega) = \pi^p(X) + k.$$

Now, we are going to show that the super-replication price only takes finite values. Let $k > 0$, by cash additivity we have $\pi^p(k) = k$. Assume there exists $X \in \mathbb{R}^\Omega$ such that $\pi(X) = -\infty$ and let $X' = 2k - X$. Since π does not take the value $+\infty$, there exists $\alpha_1 \in \mathbb{R}$ such that $\alpha > \pi(X')$. We have $\pi(X) < -\alpha$ but convexity of π implies that

$$\pi(k) \leqslant \frac{1}{2}\pi(X') + \frac{1}{2}\pi(X),$$

It is equivalent to

$$k \leqslant \frac{1}{2}\alpha - \frac{1}{2}\alpha,$$

which contradicts the fact that $k > 0$. Hence, π^p does not take the value $-\infty$ on $\mathbb{R}^\Omega$. □

Proof of Theorem 3.2. Let $\alpha : \mathbb{R}^\Omega \to \mathbb{R} \cup \{+\infty\}$ be the Legendre-Fenchel transform of π^p, that is

$$\alpha(P) := \sup_{X \in \mathbb{R}^\Omega} (P \cdot X - \pi^p(X)).$$

Observe that $\alpha(P) \in [0, \infty]$ for every $P \in \Delta(\Omega)$. To see this, observe that

$$\alpha(P) \geqslant P \cdot 0 - \pi^p(0) = 0.$$

The super-replication price π is convex, moreover, we showed that it only takes finite values and is continuous. The Fenchel-Moreau theorem states that a function is equal to its biconjugate if it is proper[2], lower semi-continuous and convex. Hence, in particular, it is equal to its biconjugate if it is convex, continuous and only takes finite values. Hence, the super-replication price is equal to

$$\pi^p(X) = \sup_{P \in \mathbb{R}^\Omega} (P \cdot X - \alpha(P)).$$

We are going to show that $\alpha(P) = +\infty$ if P is not a probability.

Assume first that $P(\omega) < 0$ for some $\omega \in \Omega$ and choose an arbitrary $n > 0$. Let $X_n = -n\mathbf{1}_\omega$ where $\mathbf{1}_\omega$ is the vector of $\mathbb{R}^\Omega$ with coordinate ω equal to 1 and 0 otherwise. Lécuyer and Martins-da Rocha (2021) show that the super-replication price π^p is monotone hence since $0 \geqslant X_n$ we have $0 \geqslant \pi(X_n)$. Moreover, by construction, $P \cdot X_n = -nP(\omega) > 0$ hence we have

$$\alpha(P) \geqslant P \cdot X_n - \pi(X_n) \geqslant P \cdot X_n,$$

that is,

$$\alpha(P) \geqslant -nP(\omega) > 0.$$

Since n can be arbitrarily large, we have $\alpha(P) = +\infty$ when $P(\omega) < 0$ for some $\omega \in \Omega$.

Now, assume that $P \geqslant 0$ and P is not a probability. First assume that $\sum_{\omega \in \Omega} P(\omega) > 1$ and choose an arbitrary $k > 0$. By definition, α satisfies

$$\alpha(P) \geqslant k \sum_{\omega \in \Omega} P(\omega) - \pi(k\mathbf{1}_\Omega).$$

Cash-additivity implies

$$\alpha(P) \geqslant k\left(\sum_{\omega \in \Omega} P(\omega) - 1\right).$$

Since k was taken arbitrarily, it implies $\alpha(P) = +\infty$ when $P \geqslant 0$ and $\sum_{\omega\Omega} P(\omega) > 1$.

Assume now that $\sum_{\omega \in \Omega} P(\omega) < 1$ and choose an arbitrary $k > 0$. Then, by definition, α satisfies

$$\alpha(P) \geqslant -k \sum_{\omega \in \Omega} P(\omega) - \pi(-k\mathbf{1}_\Omega).$$

[2] We say that a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is proper when $f(X) < +\infty$ for at least one $X \in \mathbb{R}^n$ and $f(X) > -\infty$ for all $X \in \mathbb{R}^n$.

Therefore, by cash-additivity,

$$\alpha(P) \geqslant k\left(1 - \sum_{\omega\in\Omega} P(\omega)\right).$$

Since k was taken arbitrarily, it implies $\alpha(P) = +\infty$ when $P \geqslant 0$ and $\sum_{\omega\Omega} P(\omega) < 1$.

Hence, the super-replication price satisfies

$$\pi^p(X) = \sup_{P\in\Delta(\Omega)} (\mathbb{E}_P(X) - \alpha(P)),$$

where P is a probability. Since $\Delta(\Omega)$ is a compact set and by the extreme value theorem, a continuous function attains its supremum on a compact set, we obtain that

$$\pi^p(X) = \max_{P\in\Delta(\Omega)} (\mathbb{E}_P(X) - \alpha(P)),$$

as desired.

□

Proof of Proposition 3.2. The proof of this proposition is a small variation of the proof of Föllmer and Schied (2016) Theorem 4.15. We write it for tractability. Let $X \in \mathbb{R}^\Omega$ and let $X' = X - \pi^p(X)\mathbf{1}_\Omega$, then, we have

$$\alpha_{\min}(P) \geqslant \mathbb{E}_P(X') = \mathbb{E}_P(X) - \pi^p(X),$$

for all $P \in \Delta(\Omega)$. Hence,

$$\pi^p(X) \geqslant \sup_{P\in\Delta(\Omega)} (\mathbb{E}_P(X) - \alpha_{\min}(P)).$$

For all $X \in \mathbb{R}^\Omega$, we want to construct $Q_X \in \Delta(\Omega)$ such that

$$\pi^p(X) \leqslant \mathbb{E}_{Q_X}(X) - \alpha_{\min}(Q_X). \tag{4}$$

as it would imply

$$\pi^p(X) = \max_{P\in\Delta(\Omega)} (\mathbb{E}_P(X) - \alpha_{\min}(P)).$$

We fist fix $X \in \mathbb{R}^\Omega$ such that $\pi^p(X) = 0$. The payoff X is not contained in the set

$$\mathcal{B} = \{X \in \mathbb{R}^\Omega \text{ s.t. } \pi^p(X) < 0\},$$

which is open. Hence we can separate X and $\mathcal{B}$, that is, there exists a non-zero continuous linear functional ℓ on $\mathbb{R}^\Omega$ such that

$$\ell(X) \geqslant \sup_{Y \in \mathcal{B}} \ell(Y).$$

Let $Y \leqslant 0$, by cash-additivity and monotonicity, $-\mathbf{1}_\Omega + \lambda Y \in \mathcal{B}$ for all $\lambda > 0$. Hence,

$$\ell(X) \geqslant \ell(-1_\Omega) + \lambda \ell(Y).$$

It implies $\ell(Y) \leqslant 0$. Since ℓ is non-zero, there exists $Y \in \mathbb{R}^\Omega$ such that $\ell(Y) < 0$, moreover since it is negative we have $\ell(Y^-) < 0$. Without loss of generality we can take Y such that $\max_{\omega \in \Omega} |Y(\omega)| < 1$ and we have $\ell(-\mathbf{1}_\Omega - Y^-) \leqslant 0$. It implies $\ell(-\mathbf{1}_\Omega) = \ell(-\mathbf{1}_\Omega - Y^-) + \ell(Y^-) < 0$. We let Q_X be such that

$$\mathbb{E}_{Q_X}(Y) = \frac{\ell(Y)}{\ell(\mathbf{1}_\Omega)} \text{ for all } Y \in \mathbb{R}^\Omega.$$

We have

$$\alpha_{\min}(Q_X) = \sup_{Y \in \mathcal{A}_{\pi^p}} \mathbb{E}_{Q_X}(Y) \geqslant \sup_{Y \in \mathcal{B}} \mathbb{E}_{Q_X}(Y) = \frac{1}{\ell(\mathbf{1}_\Omega)} \sup_{Y \in \mathcal{B}} \ell(Y).$$

Moreover, for all $Y \in \mathcal{A}_{\pi^p}$, we have $Y - \epsilon \in \mathcal{B}$ for all $\epsilon > 0$. Hence $\mathcal{A}_{\pi^p} - \epsilon \subseteq \mathcal{B}$ for all $\epsilon > 0$ which implies

$$\alpha_{\min}(Q_X) = \frac{1}{\ell(\mathbf{1}_\Omega)} \sup_{Y \in \mathcal{B}} \ell(Y).$$

We have

$$E_{Q_X}(X) - \alpha_{\min}(Q_X) = \frac{1}{\ell(\mathbf{1}_\Omega)}(\ell(X) - \sup_{Y \in \mathcal{B}} \ell(Y)) \geqslant 0 = \pi^p(X).$$

By cash-additivity, for every $Y \in \mathbb{R}^\Omega$, we can construct Q_Y satisfying Equation 4 following the same steps by letting $X = Y - \pi^p(Y)\mathbf{1}_\Omega$. It entails

$$\pi^p(X) = \max_{P \in \Delta(\Omega)} (\mathbb{E}_P(X) - \alpha_{\min}(P)) \quad \text{ for all } X \in \mathbb{R}^\Omega.$$

Moreover, $\alpha_{\min}$ is the minimal confidence function satisfying Equation 1. Indeed, for every α satisfying Equation 1, we have

$$\alpha(P) = \sup_{X \in \mathbb{R}^\Omega} E_P(X) - \pi^p(X) \geqslant \sup_{X \in \mathcal{A}_{\pi^p}} E_P(X) - \pi^p(X) \geqslant \alpha_{\min}(P),$$

for all $P \in \Delta(\Omega)$.

□

Proof of Theorem 3.3. We first show that if the market structure (p, G) does not admit arbitrage then there exists a payoff $X_0 \in \mathbb{R}^\Omega$ such that for all payoffs $X \in \mathbb{R}^\Omega$, we have

$$[X \geqslant 0 \quad \text{and} \quad \pi^p(X + X_0) - \pi^p(X_0) \leqslant 0] \implies X = 0.$$

First, we assume that there is no arbitrage opportunity and we are going to show that there exist a payoff $X_0 \in \mathbb{R}^\Omega$ and a probability $\mu \in \Delta(\Omega)$ such that

$$\alpha(\mu) = 0, \quad \pi^p(X_0) = \mathbb{E}_\mu(X_0),$$

and

$$\pi^p(X) \geqslant \mathbb{E}_\mu(X) \quad \text{for all} \quad X \in \mathbb{R}^\Omega.$$

.

Lécuyer and Martins-da Rocha (2021) show that a lower-semicontinuous super-replication price π^p defined on a closed set is consistent with the absence of arbitrage if, and only if there exist a payoff $X_0 \in \mathbb{R}^\Omega$ and a strictly positive vector $\mu \in \mathbb{R}^\Omega_{++}$ such that

$$\pi^p(X) - \pi^p(X_0) \geqslant \mu \cdot (X - X_0), \tag{5}$$

for all $X \in \mathbb{R}^\Omega$. We can rewrite the condition as

$$\pi^p(X + X_0) - \pi^p(X_0) \geqslant \mu \cdot X.$$

for all $X \in \mathbb{R}^\Omega$. Cash-additivity implies $\sum_{\omega\in\Omega} \mu(\omega) = 1$. Hence, μ is a probability.

Taking $X = 0$ in Equation 5, it implies $\mathbb{E}_\mu(X_0) \geqslant \pi^p(X_0)$. Denote P_{X_0} the probability such that

$$\pi^p(X_0) = \mathbb{E}_{P_{X_0}}(X_0) - \alpha(P_{X_0}).$$

We have

$$\nabla[\mathbb{E}_\mu(X_0) - \mathbb{E}_{P_{X_0}}(X_0) + \alpha(P_{X_0})] = \mu - P_{X_0} \geqslant 0.$$

It implies $\mu = P_{X_0}$. Therefore $\alpha(\mu) = 0$ and $\pi^p(X_0) = \mathbb{E}_\mu(X_0)$. It entails that

$$\pi^p(X) \geqslant \mu \cdot X \tag{6}$$

for all $X \in \mathbb{R}^\Omega$.

Conversely, assume that there exist a payoff $X_0 \in \mathbb{R}^\Omega$ and a probability $\mu \in \Delta(\Omega)$ such that

$$\alpha(\mu) = 0, \quad \pi^p(X_0) = \mathbb{E}_\mu(X_0),$$

and

$$\pi^p(X) \geqslant \mathbb{E}_\mu(X) \quad \text{for all} \quad X \in \mathbb{R}^\Omega.$$

We are going to show that there is no arbitrage opportunity. Let $X \in \mathbb{R}^\Omega$ such that

$$X \geqslant 0 \quad \text{and} \quad \pi^p(X + X_0) - \pi^p(X_0) \leqslant 0.$$

We have

$$0 \geqslant \pi^p(X + X_0) - \mathbb{E}_\mu(X_0) \geqslant \mathbb{E}_\mu(X + X_0) - \mathbb{E}_\mu(X_0) = \mathbb{E}_\mu(X).$$

Since $\mu > 0$, we have $0 \geqslant \mathbb{E}_\mu(X) \geqslant 0$. Hence $X = 0$. □

6.2 Proof of Theorem 4.1

We introduce first preliminary definitions.

Definition 6.1. A function $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ is

1 continuous if for every sequence $(X)_{n=0}^\infty$ in $\mathbb{R}^\Omega$, $\lim_n X_n = X$ implies $\lim_n \pi(X_n) = \pi(X)$;

2 disjoint exponential-additive if for every $X, Y \in \mathbb{R}^\Omega$

$$\operatorname{supp} X \cap \operatorname{supp} Y = \emptyset \implies e^{\pi(X+Y)} = e^{\pi(X)} + e^{\pi(Y)} - 1;$$

3 translation invariant if $\pi(X + k) = \pi(X) + k$;

4 normalized if $\pi(k) = k$ for every $k \in \mathbb{R}$;

5 strictly monotone if $X \geqslant Y \implies \pi(X) \geqslant \pi(Y)$ and $\pi(X) > \pi(Y)$ if it further holds that $X(k) > Y(k)$ for some k.

The next novel key lemma characterizes functionals that admit an entropic representation.

Lemma 6.1. A functional $\pi : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies (1)-(5) if and only if there exists $Q \in \Delta(\Omega)$ with full support such that

$$\pi(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}_P X - R(P||Q) = \log(\mathbb{E}_Q(e^X)),$$

Proof of Theorem 4.1.

Sufficiency. Define $\pi^e : \mathbb{R}^\Omega \to \mathbb{R}$ by $\pi^e(X) = e^{\pi(X)} - 1$ for every $X \in \mathbb{R}^\Omega$. Then conditions (1), (2), and (5) imply that π^e satisfies the conditions of Theorem 2 in Stanca (2020), so that there exists $Q \in \Delta(\Omega)$ and a continuous and strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi(0) = 0$ such that

$$\pi^e(X) = \mathbb{E}_Q \phi(X),$$

for every $X \in \mathbb{R}^\Omega$. Moreover, Q must have full support because of (5). To see this, observe that Q satisfies $Q(A) = \pi^e(\mathbf{1}_A)$ for every $A \subseteq \Omega$.

Finally, observe that since π is normalized we obtain $\pi^e(k) = \phi(k) = e^k - 1$ for every $k \in \mathbb{R}$. Hence, since $\phi^{-1}(k) = \log(k+1)$, we obtain

$$\pi(X) = \phi^{-1}(\pi^e(X)) = \phi^{-1}(\mathbb{E}_Q \phi(X)) = \log\Big(\mathbb{E}_Q\Big(e^X\Big)\Big).$$

Hnece by Proposition 1.4.2 in Dupuis and Ellis (2011), we obtain that

$$\pi(X) = \max_{P \in \Delta(\Omega)} \mathbb{E}X - R(P||Q),$$

which concludes this part of the proof.

Necessity. We only prove disjoint additivity, as checking the necessity of the other conditions is routine. Consider $X, Y \in \mathbb{R}^\Omega$ with $\operatorname{supp} X \cap \operatorname{supp} Y = \emptyset$. We have

$$\begin{aligned} e^{\pi(X+Y)} &= \mathbb{E}_Q\Big(e^{X+Y}\Big) \\ &= \mathbb{E}_Q\Big(e^{X} e^{aY}\Big) \\ &= \mathbb{E}_Q\Big(e^{X}\mathbf{1}_{\operatorname{supp} X}\Big) + \mathbb{E}_Q\Big(e^{Y}\mathbf{1}_{\operatorname{supp Y}}\Big) - 1 \\ &= e^{\pi(X)} + e^{\pi(Y)} - 1, \end{aligned}$$

as desired. □

Lemma 6.2. Assume that G is monotone, linear, $G(\mathbb{R}^J) = \mathbb{R}^\Omega$, and that p is strictly monotone and satisfies segmented additivity. Then π satisfies (1)-(6).

Proof. We show that π satisfies (2) as the checking that the other properties are satisfied is routine. Recall that the super-replication price is given by

$$\pi(X) = \inf\{p(\theta) \ : \ \theta \in \mathbb{R}^J \quad \text{and} \quad X \leqslant G(\theta)\}.$$

Take $X, Y \in \mathbb{R}^\Omega$ such that $\operatorname{supp} X \operatorname{supp} Y = \emptyset$. Since p is continuous, strictly monotone and G is linear, monotone and satisfies $G(\mathbb{R}^J) = \mathbb{R}^\Omega$ we can find portfolios $\tilde{\theta}$ and $\tilde{\theta}'$ such that

$$\operatorname{supp} \tilde{\theta} \cap \operatorname{supp} \tilde{\theta}' = \emptyset,$$

$G(\tilde{\theta}) = X$, $G(\tilde{\theta}') = Y$, $p(\tilde{\theta}) = \pi(X)$ and $p(\tilde{\theta}') = \pi(Y)$. It follows that

$$\begin{aligned} \pi(X+Y) &= \inf\{p(\theta) \ : \ \theta \in \mathbb{R}^J, X+Y \leqslant G(\theta)\} \\ &= \inf\{p(\theta+\theta') \ : \ \theta, \theta' \in \mathbb{R}^J, \operatorname{supp} \theta \cap \operatorname{supp} \theta' = \emptyset, X = G(\theta), Y = G(\theta')\}. \end{aligned}$$

Hence, because $x \mapsto e^x$ is a strictly increasing and continuous function, and since $\operatorname{supp} G(\theta) \cap \operatorname{supp} G(\theta') = \emptyset$ for any $G(\theta) = X$, $G(\theta') = Y$, by segmented additivity we obtain

$$\begin{aligned} e^{\pi(X+Y)} &= \exp\Big\{\inf\{p(\theta+\theta') \ : \ \theta, \theta' \in \mathbb{R}^J, \operatorname{supp} \theta \cap \operatorname{supp} \theta' = \emptyset, X = G(\theta), Y = G(\theta')\}\Big\} \\ &= \inf\{e^{p(\theta+\theta')} \ : \ \theta, \theta' \in \mathbb{R}^J, \operatorname{supp} \theta \cap \operatorname{supp} \theta' = \emptyset, X = G(\theta), Y = G(\theta')\} \\ &= \inf\{e^{p(\theta)} + e^{p(\theta')} - 1 \ : \ \theta, \theta' \in \mathbb{R}^J, \operatorname{supp} \theta \cap \operatorname{supp} \theta' = \emptyset, X = G(\theta), Y = G(\theta')\} \\ &= \inf\{e^{p(\theta)} + e^{p(\theta')} \ : \ \theta, \theta' \in \mathbb{R}^J, \operatorname{supp} \theta \cap \operatorname{supp} \theta' = \emptyset, X = G(\theta), Y = G(\theta')\} - 1 \\ &= \inf\{e^{p(\theta)} \ : \ \theta \in \mathbb{R}^J, X = G(\theta)\} + \inf\{e^{p(\theta')} \ : \ \theta' \in \mathbb{R}^J, Y = G(\theta')\} - 1. \end{aligned}$$

We can therefore conclude that

$$e^{\pi(X+Y)} = e^{p(\tilde{\theta})} + e^{p(\tilde{\theta}')} = e^{\pi(X)} + e^{\pi(Y)},$$

as desired. □

We can now deliver the proof of the result.

Proof of Theorem 4.1. By Lemma 6.2, π satisfies (1)-(5). Therefore the result follows by applying Lemma 6.1. □

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