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A simplified approach to subjective expected utility

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ABSTRACT

I provide a novel simplified approach to Savage's theory of subjective expected utility. Such an approach is based on abstract integral representation theorems in the space of measurable functions. The advantage of such an approach is that these results can be used to easily obtain variations on Savage's theorem, such as representations with state-dependent utility or probability measures that can have atoms. Finally, I discuss how such an approach can be used in other settings such as decision making under ambiguity.

1. Introduction

Savage's (1954, 1972) axiomatization of subjective expected utility (SEU) is the cornerstone of decision making under uncertainty. However, the predominant approach in the subsequent literature on decision making under uncertainty is the so-called Anscombe–Aumann (AA) framework. This predominance is due to its analytical tractability compared to the Savage set-up. For example, Machina and Schmeidler (1995) referring to Anscombe and Aumann's approach state

> "Since their article came out seven years after Savage's book, and was not even the first one to consider the mixed subjective/objective case [...] one may ask why it enjoys the reputation that it does. The answer is the elegance and simplicity of the Anscombe–Aumann characterization." (p. 108)

One of the advantages of the AA framework is that it enables the use of powerful techniques originating from functional-analysis. This fact allows decision theorists to develop refinements and modifications of the SEU model in the AA framework (e.g., Fishburn, 1970; Schmeidler, 1989; Gilboa and Schmeidler, 1989; Maccheroni et al., 2006). For example, SEU can be obtained in the AA set-up as a consequence of the representation theorem for linear functionals from Dunford and Schwartz (1958). The Choquet expected utility representation uses the integral representation theorem developed by Schmeidler (1986). The axiomatization of variational preferences in Maccheroni et al. (2006) built on earlier results due to Dolecki and Greco (1995).

In this paper, I show how one can obtain Savage-style representation theorems by means of a functional analytic approach as in the AA framework. I provide abstract integral representation theorems and then show how these can be used in a decision-theoretic framework. To illustrate these results, let $(\Omega, \mathcal{F})$ be a measurable space and consider a functional I on the space of simple $\mathcal{F}$-measurable functions. I provide conditions under which I can be written as[2]

$$I(f) = \int u(\omega, f(\omega))\mu(d\omega), \tag{1}$$

and

$$I(f) = \int u(f(\omega))\mu(d\omega), \tag{2}$$

for every bounded measurable function f. The key condition for these results is the following additivity condition: if f, g are such that $\{\omega : f(\omega) \neq 0\} \cap \{\omega : g(\omega) \neq 0\} = \emptyset$ then $I(f+g) = I(f) + I(g)$. In words, this assumption requires additivity restricted to functions that have disjoint support. These integral representation theorems are functional-analytic in the sense that they rely on the use of the Radon–Nykodym theorem.

The advantage of such an approach is that it can be used in axiomatic decision theory to readily obtain variations on Savage's SEU model. Using the representation in (1), I obtain a representation theorem for SEU with state-dependent utility. Moreover, using (2), I obtain a version of Savage's SEU model and show how non-atomicity of the prior can be easily relaxed. From a mathematical perspective, these functional-analytic techniques have the advantage of considerably simplifying Savage's derivation of expected utility.

Finally, I discuss how these results can be used for axiomatic decision theory in different settings. To illustrate this point, I

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[2] For expositional purposes I do not discuss here the properties that the family of functions $u(\omega, \cdot)_{\omega\in\Omega}$, and the measure μ must satisfy, nor what type of uniqueness holds in the representation. See Section 2 for details.

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provide a general axiomatization of second order expected utility (Grant et al., 2009; Strzalecki, 2011). In Klibanoff et al. (2019), the representation in (2) is applied to obtain an axiomatization of the smooth ambiguity model.

This paper is closely related to the mathematical literature on integral representation theory, which extensively examined the additivity condition discussed previously. In particular, the present paper is related to the work of Martin and Mizel (1964). The condition that allows the relaxation of non-atomicity of the probability measure is taken from their paper. When translated into preferences, such a condition is equivalent to the unlikely atoms axiom from Mackenzie (2019). Section 4.3 briefly reviews this literature in mathematics.

In decision theory, a similar approach to mine was adopted by Wakker and Zank (1999) and Castagnoli and LiCalzi (2006). Section 4.2 discusses their work and connections with the axiomatic literature in decision theory.

1.1. Structure

Section 2 presents the main abstract integral representation theorems. These are then applied in Section 3 to obtain different Savage-style representation theorems in a decision-theoretic framework. Section 3.2 contains a general axiomatization of second order expected utility. Section 4 offers a discussion of the main results with emphasis on potential further applications and a discussion of the related literature. The Appendix discusses extensions of the main results.

2. Integral representation theorems

2.1. Preliminaries

Consider a measurable space $(\Omega, \mathcal{F})$. $B_0(\mathcal{F}, K)$ denotes the set of all simple $\mathcal{F}$-measurable functions with range contained in the interval $K \subseteq \mathbb{R}$, where I assume that $0 \in K$ and that K contains a positive number (K can be unbounded, for instance $K = \mathbb{R}$ is allowed). Fix $\bar{k} \in K$ such that $\bar{k} > 0$. $B(\mathcal{F}, K)$ is the set of all bounded $\mathcal{F}$-measurable real valued functions with range contained in K.[3] $\Delta(\Omega)$ is the set of all countably additive probability measures defined on $\mathcal{F}$, while $\Delta(\Omega)_{na} \subseteq \Delta(\Omega)$ is the set of probability measures that are non-atomic, i.e. $\mu \in \Delta_{na}(\Omega)$ if for every $A \in \mathcal{F}$ such that $\mu(A) > 0$, there exists $B \in \mathcal{F}$ such that $B \subseteq A$ and $\mu(A) > \mu(B) > 0$. If $\mu, \lambda \in \Delta(\Omega)$, write $\mu \ll \lambda$ if μ is absolutely continuous with respect to λ, i.e. for every $A \in \mathcal{F}$, $\lambda(A) = 0 \implies \mu(A) = 0$. Say that two measures μ and λ are mutually absolutely continuous if $\mu \ll \lambda$ and $\lambda \ll \mu$. A constant $k \in K$ denotes the constant function that takes the value k for every $\omega \in \Omega$. As usual, a simple function $f \in B_0(\mathcal{F}, K)$ will be written as $f = \sum_{i=1}^{n} x_i 1_{A_i}$ where $(A_i)_{i=1}^{n}$ is a partition of Ω. For a singleton $\{\omega\}$, for simplicity write 1_ω in place of $1_{\{\omega\}}$. For $A \in \mathcal{F}$ and $f \in B_0(\mathcal{F}, K)$, $f1_A$ denotes a function that takes the value $f(\omega)$ for $\omega \in A$ and 0 for $\omega \notin A$. For any $f \in B_0(\mathcal{F}, K)$, let supp$f$ denote the support of the function f, i.e. $\text{supp} f = \{\omega \in \Omega : f(\omega) \neq 0\}$. A functional $I : B_0(\mathcal{F}, K) \to \mathbb{R}$ is:

1 *non-negative* if $I(f) \geq 0$ for every $f \in B_0(\mathcal{F}, K)$;
2 *normalized* if $I(\bar{k}) = 1$;
3 *monotone* if $f \geq g$ implies $I(f) \geq I(g)$ and *strictly monotone* if for every $A \in \mathcal{F}$ and $f, g \in B_0(\mathcal{F}, K)$ such that $I(\bar{k}1_A) > 0$, $f(\omega) > g(\omega)$ for every $\omega \in A$ implies $I(f1_A) > I(g1_A)$;
4 *disjoint additive* if for every $f, g \in B_0(\mathcal{F}, K)$,
$$\text{supp} f \cap \text{supp} g = \emptyset \implies I(f + g) = I(f) + I(g);$$
5 *weakly scale-invariant* if for every $A \in \mathcal{F}$ and every $x \in K$,
$$I(\bar{k}1_A) = 0 \implies I(x1_A) = 0;$$
6 *strongly scale-invariant* if for every $A, B \in \mathcal{F}$ and $x \in K$,
$$I(\bar{k}1_A) = I(\bar{k}1_B) \implies I(x1_A) = I(x1_B);$$
7 *continuous* if for every sequence $(f_n)_{n=1}^{\infty}$ such that f_n converges to f pointwise and there exists $m, M \in \mathbb{R}$ such that $m \leq f_n(\omega) \leq M$ for every $\omega \in \Omega$ then $I(f_n) \to I(f)$;
8 *non-atomic* if for every finite set $A \in \mathcal{F}$ and $x \in K$ it holds that $I(x1_A) = 0$.

Properties 1–3 are standard. Property 4 is the key condition of this paper. It implies that if two sets $A, B \in \mathcal{F}$ are disjoint, then $I(x1_A + x1_B) = I(x1_A) + I(x1_B)$ for any $x \in K$. In particular, this condition will allow the measure μ to be defined by $\mu(A) = I(\bar{k}1_A)$ for every $A \in \mathcal{F}$. This fact will be key for all the results in this paper. Property 5 says that if the functional is zero at $\bar{k}1_A$ then it will remain zero for any rescaling $x1_A$ of $\bar{k}1_A$. Likewise, property 6 states that if I assigns the same value to $\bar{k}1_A$ and $\bar{k}1_B$ then it will also assign the same value to any rescaling of the two functions. Note that if $I(0) = 0$ (which will be true if property 4 holds), then 6 effectively implies 5. Moreover, observe that both properties of scale invariance are implied by homogeneity of degree 1 (i.e., $I(\alpha f) = \alpha I(f)$ for every $\alpha \in \mathbb{R}$ and function f), but clearly neither of them implies homogeneity. Property 7 is a continuity assumption reminiscent of the bounded convergence theorem for integrals. Property 8 will be key to guarantee that the measure defined by $A \mapsto I(\bar{k}1_A)$ is non-atomic.

2.2. Main results

The first result characterizes functionals that are non-negative, normalized, disjoint additive, weakly scale-invariant and continuous.

Theorem 1. *$I : B_0(\mathcal{F}, K) \to \mathbb{R}$ is non-negative, normalized, disjoint additive, weakly scale invariant and continuous if and only if there exists $u : \Omega \times \mathbb{R} \to \mathbb{R}$ with $u(\cdot, x)$ measurable for all $x \in \mathbb{R}$, $u(\cdot, x) \geq 0$ μ-a.s. $\forall x \in \mathbb{R}$, $u(\omega, 0) = 0$, $u(\omega, \bar{k}) = 1$ for every $\omega \in \Omega$ and $\mu \in \Delta(\Omega)$ such that*
$$I(f) = \int u(\omega, f(\omega))\mu(d\omega), \tag{3}$$
where the mapping $f \mapsto \int u(\omega, f(\omega))\mu(d\omega)$ is continuous.

Proof. I only prove the necessity of disjoint additivity (it is straightforward to check the other conditions). If $f, g \in B_0(\mathcal{F}, K)$ satisfy $\text{supp} f \cap \text{supp} g = \emptyset$, then there exist a measurable partition $(A_i)_{i=1}^{N+M}$ of Ω such that $f = \sum_{i=1}^{N} x_i 1_{A_i}$, $g = \sum_{i=N+1}^{N+M} x_i 1_{A_i}$ and $f + g = \sum_{i=1}^{N+M} x_i 1_{A_i}$. Therefore,
$$\begin{aligned}
I(f + g) &= \int u(\omega, f(\omega) + g(\omega))\mu(d\omega) = \int \sum_{i=1}^{N+M} u(\omega, x_i)1_{A_i}\mu(d\omega) \\
&= \sum_{i=1}^{N+M} \int_{A_i} u(\omega, x_i)\mu(d\omega) \\
&= \sum_{i=1}^{N} \int_{A_i} u(\omega, x_i)\mu(d\omega) + \sum_{i=N+1}^{N+M} \int_{A_i} u(\omega, x_i)\mu(d\omega) \\
&= \int u(\omega, f(\omega))\mu(d\omega) + \int u(\omega, g(\omega))\mu(d\omega),
\end{aligned}$$
as desired.

To show that the conditions are sufficient for the representation, define $\mu : \mathcal{F} \to \mathbb{R}$ by $\mu(E) = I(\bar{k}1_E)$ for every $E \in \mathcal{F}$. I

[3] See Proposition 2 in the Appendix for a generalization of the main results to vector-valued functions.

claim that $\mu \in \Delta(\Omega)$. To see this, note that since I is non-negative $\mu(A) \geq 0$ for every $A \in \mathcal{F}$. By disjoint additivity, for any $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, $\mu(A \cup B) = I(\bar{k}1_A + \bar{k}1_B) = I(\bar{k}1_A) + I(\bar{k}1_B) = \mu(A) + \mu(B)$. Moreover, disjoint additivity implies $I(0) = 0$ so that $\mu(\emptyset) = 0$. Finally, take any decreasing sequence of measurable sets $(A_i)_{i=1}^{\infty}$ such that $\cap_{i=1}^{\infty} A_i = \emptyset$. The sequence $(\bar{k}1_{A_i})_{i=1}^{\infty}$ is uniformly bounded and converges pointwise to 0. By continuity, $\lim_{n\to\infty} \mu(\cap_{i=1}^{n} A_i) = \mu(A_n) = \lim_{n\to\infty} I(\bar{k}1_{A_n}) = I(0) = 0$.

Now define the measure υ_x by $\nu_x(E) = I(x1_E)$ for every $x \in K$. Using the same reasoning as above, we find that ν_x is a countably additive measure. Moreover, since I is weakly scale invariant, ν_x is μ-absolutely continuous for every $x \in K$. By the Radon–Nykodym theorem (e.g., see Billingsley, 2008 Theorem 32.2) it follows that there exist functions $(g_x)_{x\in K}$, all measurable such that $I(x1_E) = \nu_x(E) = \int_E g_x(\omega)\mu(d\omega)$. Moreover, $g_x \geq 0$ μ-a.s for every $x \in K$ and $g_0 = 0$, $g_{\bar{k}} = 1$. But then letting $u(\omega, x) = g_x(\omega) \quad \forall \omega \in \Omega$, for any $f \in B_0(\mathcal{F}, K)$ we get

$$I(f) = I\left(\sum_i x_i 1_{A_i}\right) = \sum_i I(x_i 1_{A_i})$$

$$= \sum_i \int_{A_i} g_{x_i}\mu = \int \sum_i 1_{A_i} g_{x_i}(\omega)\mu(d\omega) = \int u(\omega, f(\omega))\mu(d\omega),$$

as desired. □

The uniqueness of this representation theorem is weak, as illustrated by the next result.

Proposition 1. *There exist $\mu' \in \Delta(\Omega)$ and $u' : S \times K \to \mathbb{R}$ with $u'(\cdot, x)$ measurable for every $x \in K$ such that $I(f) = \int u'(\omega, f(\omega))\mu'(d\omega)$ for every $f \in B_0(\mathcal{F}, K)$ if and only if $\mu \ll \mu'$ and $u'(\omega, x) = u(\omega, x)\frac{d\mu'}{d\mu}(\omega)$ μ'-a.s.*

Proof. See the Appendix. □

In the proof of Theorem 1 the fact that set function defined by $\mu(A) = I(\bar{k}1_A)$ is a measure relies on non-negativity of I. The next integral representation theorem takes a different approach by dropping non-negativity and imposing monotonicity, which along with the other conditions will imply that μ is a measure. Moreover, to obtain uniqueness of the representation I will also require two additional conditions: strong scale-invariance and non-atomicity.

As in Kopylov (2010), say that $\mathcal{F}$ is countably separated if it contains a countable collection of events $\mathcal{C} \subseteq \mathcal{F}$ such that for any $s, s' \in S$, there is $E \in \mathcal{C}$ such that $s \in E$ and $s \notin E$.

Theorem 2. *Suppose that $\mathcal{F}$ is countably separated. $I : B_0(\mathcal{F}, K) \to \mathbb{R}$ is normalized, disjoint additive, monotone, strongly scale invariant, continuous and non-atomic if and only if there exist $u : \mathbb{R} \to \mathbb{R}$ with $u(0) = 0$ and $u(\bar{k}) = 1$, continuous and non-decreasing, $\mu \in \Delta_{na}(\Omega)$ such that*

$$I(f) = \int u(f(\omega))\mu(d\omega).$$

Moreover, u is strictly increasing if and only if I is strictly monotone. Finally, the representation is unique.

Proof. It is routine to check the necessity of all the conditions.

As for sufficiency, again define $\mu : \mathcal{F} \to \mathbb{R}$ by

$$\mu(A) = I(\bar{k}1_A) \quad \forall A \in \mathcal{F}.$$

Note that since $\bar{k} \geq \bar{k}1_A \geq 0$, by monotonicity we have $1 = I(\bar{k}) = \mu(\Omega) \geq \mu(A) = I(\bar{k}1_A) \geq I(0) = 0$, so that $0 \leq \mu \leq 1$ and $\mu(\Omega) = 1$. The same reasoning as in the proof of Theorem 1 can now be used to show that disjoint additivity and continuity imply that μ is countably additive. Finally, since I is non-atomic, it follows that $\mu(\{\omega\}) = 0$ for every $\omega \in \Omega$. Thus by Lemma 8 in Kopylov (2010) it follows that $\mu \in \Delta_{na}(\Omega)$. Now for every $a \in K$, define $\varphi_a : \mathcal{F} \to \mathbb{R}$ by

$$\varphi_a(A) = I(a1_A) \quad \forall A \in \mathcal{F}.$$

Note that $\phi_a \ll \mu$. To see this, note that by strong scale-invariance and disjoint additivity if $\mu(A) = 0$, then $\mu(A) = I(\bar{k}1_A) = I(1_\emptyset) = 0 \implies I(a1_A) = I(a1_\emptyset) = 0$. Now similar to the proof of Theorem 1.6 in Martin and Mizel (1964), I claim there exists $\rho : (0, 1] \to \mathbb{R}$ such that

$$\varphi_a(A) = \rho(\mu(A))\mu(A) \quad \forall A \in \mathcal{F}. \tag{4}$$

To prove this claim, for any $s \in [0, 1]$ let $\mathcal{Q}(s) = \{A \in \mathcal{F} : \mu(A) = s\}$. Since $\mu \in \Delta_{na}(\Omega)$, $\mathcal{Q}(s) \neq \emptyset$ for every $s \in [0, 1]$. Let $s \in (0, 1]$. Then for any $A, B \in \mathcal{Q}(s)$, strong scale-invariance implies $\varphi_a(A) = \varphi_a(B)$ so that

$$\frac{\varphi_a(A)}{\mu(A)} = \frac{\varphi_a(B)}{\mu(B)},$$

from which (4) follows. Now I claim that ρ is (i) continuous and (ii) satisfies the functional equation

$$(s+t)\rho(s+t) = s\rho(s) + t\rho(t) \quad \forall s, t \in (0, 1] \text{ such that } s+t \in (0, 1].$$

To see (i), without loss of generality consider a sequence $(s_n)_n$ in $(0, 1]$ such that $s_n \uparrow s \in (0, 1]$. Then because $\mu \in \Delta_{na}(\Omega)$, by using Theorem 15 in Fryszkowski (2004) one can find an increasing sequence $(S_n)_n$ of sets such that $\mu(S_n) = s_n$ for every n. Thus, letting $S = \lim_{n\to\infty} S_n$, we get $\mu(S_n) \to \mu(S) = s$. Since $\phi_a \ll \mu$, it follows that $\phi_a(S_n) \to \phi(S)$. Hence $\rho(s_n) = \frac{\phi_a(S_n)}{\mu(S_n)} \to \frac{\phi_a(S)}{\mu(S)} = \rho(s)$. As for (ii), note that for any $s, t \in (0, 1]$ such that $s + t \in (0, 1]$ it holds $t \leq 1 - s = \mu(\Omega \setminus S)$ where $S \in \mathcal{Q}(s)$. Thus by non-atomicity one can find $T \subseteq \Omega \setminus S$ such that $\mu(T) = t$. Therefore,

$$\begin{aligned}(s+t)\rho(s+t) &= (\mu(S) + \mu(T))\frac{\phi_a(S \cup T)}{\mu(S \cup T)} \\ &= \phi_a(S) + \phi_a(T) = s\rho(s) + t\rho(t).\end{aligned}$$

These two claims imply that ρ is constant. Indeed, for any $s \in (0, 1]$ and natural number n it holds $s\rho(\frac{s}{n}) = s\rho(s) \implies \rho(\frac{s}{n}) = \rho(s)$. By the same reasoning, if $\frac{m}{n} \in (0, 1]$, then $\rho(\frac{m}{n}s) = \rho(s)$ for every $s \in (0, 1]$. By continuity of ρ it follows that $\rho(t) = \rho(s)$ for every $s, t \in (0, 1]$.

Now define $u : K \to \mathbb{R}$ by $u(a) = I(a)$. By monotonicity and continuity, u is continuous and non-decreasing. Also note that

$$u(a) = I(a) = \frac{\phi_a(\Omega)}{\mu(\Omega)} = \rho_a.$$

Now for any $f \in B_0(\mathcal{F}, K)$ we get

$$I(f) = I\left(\sum_{i=1}^{n} 1_{A_i} x_i\right) = I\left(1_{A_i} x_i\right) =$$

$$= \sum_{i=1}^{n} \phi_{x_i}(A_i) = \sum_{i=1}^{n} \rho_{x_i}\mu(A_i) = \sum_{i=1}^{n} u(x_i)\mu(A_i) = \int u(f(s))\mu(ds),$$

as desired. Finally, it is straightforward to check that u is strictly increasing if and only if I is strictly monotone. To see why uniqueness holds, consider $u' : K \to \mathbb{R}$ and $\mu' \in \Delta(\Omega)$ such that $I(f) = \int u'(f(\omega))\mu'(d\omega)$ for every $f \in B_0(\mathcal{F}, K)$. First note that for every $x \in K$ it holds that $u'(x) = I(x) = u(x)$, so that in particular $u(0) = u'(0) = 0$ and $u(\bar{k}) = u'(\bar{k}) = 1$. This implies that for every $A \in \mathcal{F}$, $\mu'(A) = I(\bar{k}1_A) = \mu(A)$ so that $\mu' = \mu$ as desired. □

This representation can be easily extended to $B(\mathcal{F}, K)$, whenever K is closed. For any $f \in B(\mathcal{F}, K)$, take a sequence $(f_n)_n$ that converges to f uniformly. Since $u(f_n(\omega)) \to u(f(\omega))$ for every $\omega \in \Omega$ and because $m \le f_n \le M$ for some constants m, M, since u is non-decreasing we obtain $m \le u(f_n) \le M$. Thus by the dominated convergence theorem

$$I(f) = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} \int u(f_n(\omega))\mu(d\omega) = \int u(f(\omega))\mu(d\omega),$$

as desired. Furthermore, an extension to arbitrary measurable functions could be obtained using the same techniques used by Wakker (1993).

From a mathematical perspective, Theorem 2 is a generalization of Theorem 1.6 in Martin and Mizel (1964).[4] They obtain the same representation except they start from a given measure space $(\Omega, \mathcal{F}, \mu)$ and study functionals defined on the set of bounded measurable functions, i.e. $I : L^\infty(\Omega, \mathcal{F}, \mu) \to \mathbb{R}$, thus starting with a given measure μ (also assuming $K = \mathbb{R}$). In their representation theorem, they derive a unique continuous function $u : \mathbb{R} \to \mathbb{R}$ such that

$$I(f) = \int u(f(s))\mu(ds).$$

It should be noted that this approach by no means relies on $\mathcal{F}$ being countably separated.[5] Furthermore, Proposition 2 in the Appendix extends Theorem 2 in several ways (for example, u does not have to be non-decreasing).

I now discuss how to relax the non-atomicity of the probability μ. Using the same techniques as in Martin and Mizel (1964) (see Theorem 1.11 in their paper) it is possible to allow μ to have atoms. Their result is based on the following assumption. Given the measure space $(\Omega, \mathcal{F}, \mu)$, by decomposing μ as

$$\mu = \mu_\infty + \mu_0,$$

where μ_∞ and μ_0 are the non-atomic and atomic part, it has to hold that $\mu_\infty(\Omega) \ge \mu_0(\Omega)$. To translate such a condition in the current setting, let

$$A = \{\omega \in \Omega : I(\bar{k}1_\omega) > 0\}.$$

The assumption of non-atomicity of I can be weakened by requiring that $I(\bar{k}1_A) \le \frac{1}{2}$. Observe that if I admits a representation as in Theorem 2, then $I(\bar{k}1_A) = 0$.

Theorem 3. *Suppose that $\mathcal{F}$ is countably separated. $I : B_0(\Sigma, K) \to \mathbb{R}$ is a normalized, disjoint additive, monotone, strongly scale invariant, continuous and satisfies $I(\bar{k}1_A) \le \frac{1}{2}$ if and only if there exist a continuous and non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ with $u(0) = 0$ and $u(\bar{k}) = 1$, $\mu \in \Delta(\Omega)$ such that $\mu(A) \le \frac{1}{2}$ and*

$$I(f) = \int u(f(\omega))\mu(d\omega).$$

Moreover, u is strictly increasing if and only if I is strictly monotone. Finally, the representation is unique.

Proof. First I claim that A is a countable subset of Ω. By contradiction, assume that A is uncountable. Then by letting $A_n = \{\omega \in \Omega : I(\bar{k}1_\omega > \frac{1}{n})\}$, it must be that $A = \cup_{n\ge 1} A_n$. It follows that there must be an N such that A_N is uncountable. Hence we can pick elements $\omega_1, \dots, \omega_M$ from A_N with $M > N$ so that by monotonicity $I(\bar{k}) \ge I(\bar{k}1_{\cup_{i=1}^M \{\omega_i\}}) = \sum_{i=1}^M I(\bar{k}1_{\omega_i}) > \frac{M}{N} > 1$, a contradiction.

Now define $\mu : \mathcal{F} \to \mathbb{R}$ by $\mu(E) = I(\bar{k}1_E)$ for every $E \in \mathcal{F}$. Given the previous claim, we can find an enumeration $(\omega_i)_{i=1}^\infty$ of the atoms of μ. Now we replicate the arguments in Martin and Mizel (1964), Theorem 1.8. Let $C = \Omega \setminus A$. Note that by disjoint additivity for every $f \in B_0(\mathcal{F}, K)$ it holds that

$$I(f) = I(f1_A + f1_C) = I(f1_A) + I(f1_C).$$

Observe that the functional defined by

$$f \mapsto I(f1_C),$$

satisfies all the assumptions of Theorem 2 on the measurable space $(C, \mathcal{F} \cap C)$ where $\mathcal{F} \cap C = \{E \cap C : E \in \mathcal{F}\}$. Therefore, there exists $\mu' \in \Delta_{na}(C)$ with $\mu'(E) = \mu(E)$ for every $E \in \mathcal{F} \cap C$ and $u : K \to \mathbb{R}$ continuous and non-decreasing with $u(0) = 0$ and $u(\bar{k}) = 1$ such that for every $f \in B_0(\mathcal{F}, K)$

$$I(f1_C) = \int_C u(f(\omega))\mu'(d\omega) = \int_C u(f(\omega))\mu(d\omega).$$

Now for each atom ω_i in A, define $u_i : K \to \mathbb{R}$ by $u_i(x) = \frac{I(x_i 1_{\omega_i})}{I(\bar{k}1_{\omega_i})}$ for every $x \in K$. For every $f \in B_0(\mathcal{F}, K)$, there exist atoms $(\omega_j)_{j=1}^\infty$ in A such that $f = \sum_{j=1}^\infty x_j 1_{\omega_j}$. By continuity and disjoint additivity we have

$$I(f1_A) = \sum_{i=j}^\infty I(x_j 1_{\omega_j}) = \sum_{j=1}^\infty u_j(x_i)\mu(\omega_j).$$

I now claim that $u_i = u$ for every i. This would conclude the proof since it would follow that

$$\begin{aligned} I(f) &= I(f1_A + f1_C) = I(f1_A) + I(f1_C) \\ &= \sum_{i=1}^\infty u(x_i)\mu(\omega_i) + \int_C u(f(\omega))\mu(d\omega) \\ &= \int_A u(f(\omega))\mu(d\omega) + \int_C u(f(\omega))\mu(d\omega) = \int u(f(\omega))\mu(d\omega), \end{aligned}$$

as desired. To prove the claim, first note that $I(\bar{k}1_A) \le \frac{1}{2}$ implies $\mu(A) \le \mu(C)$. Since $\mu \in \Delta_{na}(C)$, for every ω_i there is $C_i \in \mathcal{F} \cap C$ such that $\mu(C_i) = \mu(\omega_i)$. Therefore strong scale-invariance implies that

$$I(x1_{C_i}) = I(x1_{\omega_i}),$$

for every $x \in K$. It follows that

$$I(x1_{C_i}) = u(x)\mu(C_i) = I(x1_{\omega_i}) = u_i(x)\mu(\omega_i) \implies u(x) = u_i(x),$$

for every atom ω_i and $x \in K$ as desired. □

3. Applications to decision theory

In this section, the integral representation theorems just proved are applied to axiomatic decision theory. First I consider the usual Savage framework with real-valued bets. Then I consider an application to decision making under ambiguity. In the usual decision theoretic framework, $\Omega \equiv S$ is the state space, endowed with σ-algebra $\mathcal{F} \equiv \Sigma$ of events. $K \equiv X$ is the set of monetary prizes, so that $X \subset \mathbb{R}$ is an interval that contains 0 and a positive number. An act f is a simple function $f : S \to X$, so that the set of acts F can be identified with $B_0(\Sigma, X)$. fAg denotes the act that equals $f(s)$ for $s \in A$ and $g(s)$ for $s \notin A$. As usual, constant acts are identified by elements of X.

In the AA framework, the set of outcomes is the set $\Delta_s(Z)$ of simple lotteries over an arbitrary set Z and the set of AA acts is

$$F_{AA} = \{f \in \Delta_s(Z)^S : f \text{ is measurable w.r.t. } \Sigma \text{ and } |f(S)| < \infty\}.$$

A functional V defined on either F or F_{AA} represents $\succcurlyeq$ if

$$V(f) \ge V(g) \iff f \succcurlyeq g,$$

for all acts f, g.

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[4] With a different proof strategy, Cerreia-Vioglio et al. (2011b, Lemma 5.2), re-discover Martin and Mizel's theorem.

[5] The same result could be obtained by assuming the following condition on I: there is no $A \in \mathcal{F}$ such that $I(\bar{k}1_A) > 0$ and $I(\bar{k}1_B) = 0$ for every $B \in \mathcal{F}$ such that $B \subseteq A$.

3.1. Expected state-dependent utility

Consider a preference $\succcurlyeq$ over F, the set of Savage acts. By using Theorem 1, the first result of this section provides a version of subjective expected utility with state-dependent utility. Consider the following seven axioms.

P1 $\succcurlyeq$ is complete and transitive.

P2 For every $f, g, h, h' \in F$ and $E \in \Sigma$,

$$fEh \succcurlyeq gEh \implies fEh' \succcurlyeq gEh'.$$

P1–P2 are the same as Savage's. The next axiom requires the constant act that pays 0 to be the worst act.[6]

Worst outcome $f \succcurlyeq 0$, $\forall f \in F$ with $\bar{y} \succ 0$ for some constant act $\bar{y} \in X$, $\bar{y} > 0$.

Given this axiom, say that an event $A \in \Sigma$ is null if $\bar{y}_A 0 \sim 0$. Note that this notion of null event is weaker than Savage's. The next condition requires null events to have no impact on preference over acts.

P3′ $A \in \mathcal{A}$ is null $\implies xAf \sim yAf$ for all $x, y \in X$ and $f \in F$.

Contrary to Savage's P6, here the richness required is only that there exist three disjoint non-null events.

P6′ There exist at least three disjoint non-null events A_1, A_2, A_3.

The last two axioms are regularity conditions.

Certainty equivalent For every $f \in F$ there exists $x \in X$ such that $f \sim x$.

Given a sequence $(f)_{n=1}^{\infty}$ of acts, say that f_n converges preference-wise to $f \in F$ if for every act g, $g \succ f$ implies that there exists N such that $n \geq N \implies g \succ f_n$ and $f \succ g$ implies that there exists N' such that $n \geq N' \implies f_n \succ g$. .

Continuity If a sequence of acts $(f)_{n=1}^{\infty}$ satisfies $|f_n(s) - f(s)| \to 0$ for every $s \in S$ and for some m, M it holds $m \leq f_n \leq M$ for every n, then f_n converges preference-wise to f.

Theorem 4. *$\succcurlyeq$ satisfies P1, P2, P3′, P6′, worst outcome, certainty equivalent and continuity if and only if there exists $\mu \in \Delta(S)$ with $\mu(A_i) > 0$, $i = 1, \ldots, 3$, $u : S \times X \to \mathbb{R}$ with $u(\cdot, x)$ measurable for every $x \in X$, such that $\succcurlyeq$ is represented by a continuous functional $V : F \to \mathbb{R}$ such that $V(X) = V(F)$ and*

$$V(f) = \int u(s, f(s))\mu(ds),$$

where $u(s, x) \geq 0$ μ-a.s., $u(s, 0) = 0$ and $u(s, \bar{y}) = 1$ Moreover, any other $\mu' \in \Delta(\Omega)$ and $u' : S \times K \to \mathbb{R}$ represent $\succcurlyeq$ if and only if $\mu \ll \mu'$ and $u'(s, x) = a\frac{d\mu'}{d\mu}(s)u(s, x) + b(s)$ μ'-a.s. where $b : S \to \mathbb{R}$ is such that $\int b(s)\mu(ds) < \infty$.

The main step in the proof consists of applying Debreu's 1959 representation theorem to obtain a disjoint additive functional $V : F \to \mathbb{R}$ that represents $\succcurlyeq$. The result follows then by checking that all the conditions from Theorem 1 are satisfied.

Proof. It is routine to check that all the axioms are necessary.

To show sufficiency, note that by P1, P2, P6′, worst outcome, certainty equivalent and continuity we can apply Lemma 3 in the Appendix and obtain a disjoint additive and continuous functional $V : F \to \mathbb{R}$ that represents $\succcurlyeq$ with $V(0) = 0$ and $V(F) = V(X)$. By the worst outcome axiom, we can normalize V so that $V(\bar{y}) = 1$ and $V(f) \geq V(0) = 0$ for every $f \in F$. Moreover, any other functional $W : F \to \mathbb{R}$ with the same properties satisfies $W = aV + b$, for some constants a, b with $a > 0$.

Now I claim that V is weakly scale-invariant. Suppose that $V(\bar{y}1_A) = 0$. It follows that $\bar{y}A0 \sim 0$, i.e. A is null. By P3' we get that $xAf \sim 0$ for every $x \in X$. This implies that $V(x1_A) = V(0) = 0$ for every $x \in X$.

Hence by (1), we find that there exist $\mu \in \Delta(S)$, $u : S \times X \to \mathbb{R}$ with $u(\cdot, x)$ measurable for every $x \in X$, $u(s, x) \geq 0$ μ-a.s., $u(s, 0) = 0$ and $u(s, \bar{y}) = 1$ such that

$$V(f) = \int u(s, f(s))\mu(ds) \quad \text{for every } f \in F.$$

Moreover, since A_1, A_2, A_3 are non-null it follows that $V(\bar{y}1_{A_i}) > 0$ which implies that $\mu(A_i) > 0$ for $i = 1, 2, 3$.

Now suppose that $\succcurlyeq$ is represented by a functional $W(f) = \int u'(s, f(s))\mu'(ds)$. Then by uniqueness $W = aV + b$. Thus $\frac{W-b}{a} = V$, i.e.

$$\frac{W(f) - b}{a} = \int \frac{u'(s, f(s)) - b(s)}{a}\mu'(ds) = \int u(s, f(s))\mu(ds),$$

where $b : S \to \mathbb{R}$ is such that $\int b(s)\mu'(ds) = b$. By Theorem 1, $\mu \ll \mu'$ and

$$u'(s, x) = a\frac{d\mu'}{d\mu}u(s, x) + b(s),$$

μ'-a.s. desired. The converse is straightforward. □

This representation allows for quadratic utility as illustrated in the next example.

Example 1. Suppose that $S = \mathbb{R}$ endowed with Borel σ-algebra, $\mu \in \Delta(S)$ has a normal distribution, i.e. $\mu(A) = \frac{1}{\sqrt{2\pi}}\int 1_A(x)e^{-x^2/2}dx$, $K = \mathbb{R}$ and

$$u(s, x) = \begin{cases} 1 & \text{if } x = 1, \\ x^2 s^2 & \text{else .} \end{cases}$$

Then the preference induced by $V(f) = \int u(s, f(s))\mu(ds)$ for every $f \in F$ satisfies all the axioms required in Theorem 4.

Under the assumption that $X = \mathbb{R}$, Theorem 12 in Wakker and Zank (1999) provides axioms that imply that $\succcurlyeq$ is represented by a continuous function functional $V : F \to \mathbb{R}$ such that

$$V(f) = \int u(s, f(s))\mu(ds),$$

where $u(s, \cdot)$ is increasing and continuous for every $s \in S$ and μ is a non-atomic prior. Their result excludes the preferences described in the previous example, but allows for other types of state-dependent utility.

3.2. Savage-style representations

The next results use Theorems 2 and 3 to obtain a version of Savage (1972) with a probability measure that can have atoms.

As in the standard Savage setting, say that A is null if $fAg \sim g$ for every $f, g \in F$. Consider the following axioms on $\succcurlyeq$.

Strong monotonicity For any $A \in \Sigma$ non-null

$$x > y \iff xAf \succ yAg.$$

Strong monotonicity is the same as Savage's P3 except $x \succ y$ is identified with $x > y$. As a consequence of this axiom we can fix $\bar{x} \in X$ such that $\bar{x} > 0$ and $\bar{x} \succ 0$.

P4 For every $A, B \in \Sigma$ and $x, y, x', y' \in X$ such that $x \succ y$, $x' \succ y'$it holds

$$xAy \succcurlyeq xBy \implies x'Ay' \succcurlyeq x'By'.$$

P6 For every $f, g \in F$ and $x \in X$ such that $g \succ f$, there exists a Σ-measurable partition $(A_i)_{i=1}^{n}$ of S such that for every $i = 1, \ldots, n$, $g \succ xA_if$ and $xA_ig \succ f$.

P4 and P6 are exactly the same as used by Savage.

[6] The axiom could be generalized by requiring that there exists a worst act (possibly different from the constant zero act). This would require an extension of Theorem 1 similar to Proposition 2 in the Appendix.

Theorem 5. *Assume that Σ is countably separated. $\succcurlyeq$ satisfies P1,P2,P4,P6, strong monotonicity and continuity if and only if there exist $\mu \in \Delta_{na}(S)$, $u : X \to \mathbb{R}$ continuous, strictly increasing and with $u(0) = 0$, $u(\bar{x}) = 1$, such that $\succcurlyeq$ is represented by*

$$V(f) = \int u(f(s))\mu(ds).$$

Moreover, $V'(f) = \int u'(f(s))\mu'(ds)$ represents $\succcurlyeq$ if and only if $\mu = \mu'$ and there exist $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$ such that $u' = au + b$.

Proof. It is routine to check that all the axioms are necessary. As for sufficiency, note that by Lemma 1 in the Appendix, P6 implies that there exists a partition of S with three non-null events. Moreover, by strong monotonicity and continuity Lemma 2 in the Appendix implies that for every $f \in F$ there exists $x \in X$ such that $f \sim x$. Using P2 and continuity, Lemma 3 in the Appendix implies that there exists a disjoint additive and continuous functional $V : F \to \mathbb{R}$ such that $V(\bar{x}) = 1$ that represents $\succcurlyeq$. By strong monotonicity it is straightforward to verify that V is strictly monotone. By P4 V is strongly scale-invariant. Indeed, if $V(\bar{x}1_A) = V(\bar{x}1_B)$, then $\bar{x}A0 \sim \bar{x}B0$ which by P4 implies that $xA0 \sim xB0$ for any $x \in X$. By P6, the event $\{s\}$ is null for every $s \in S$, which implies that V is non-atomic. Thus by Theorem 2 it follows that there exist $u : X \to \mathbb{R}$ and $\mu \in \Delta_{na}(S)$ such that $V = \int u(f(s))\mu(ds)$.

As for uniqueness, if there exist $u' : X \to \mathbb{R}$ and $\mu' \in \Delta(S)$ such that $f \mapsto \int u'(f(s))\mu'(ds)$ represents $\succcurlyeq$, then there exist $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$ such that

$$a \int u(f(s))\mu(ds) + b = \int u'(f(s))\mu'(ds).$$

It follows that $\int \frac{(u'(f(s))-b)}{a}\mu'(ds) = \int u(f(s))\mu(ds)$. By uniqueness in Theorem 2 it follows that $u = \frac{u'-b}{a}$ and $\mu' = \mu$. Thus $u' = au+b$ as desired. The converse is straightforward. □

Thanks to Theorem 6, the non-atomicity of μ in the representation can be relaxed. As in Section 2, let

$$A = \{s \in\subseteq S : \{s\} \text{ is not null}\}.$$

The condition that requires $I(\bar{k}1_A) \leq \frac{1}{2}$ in Theorem 3 can be translated to the following axiom that is equivalent to the one introduced by Mackenzie (2019). In words, this axiom restricts the likelihood of the set of atoms.

Unlikely atoms There exist $x, y \in X$ with $x > y$ such that $xS \setminus Ay \succsim xAy$.

Theorem 6. *Assume that Σ is countably separated. $\succcurlyeq$ satisfies P1–P5, strong monotonicity, unlikely atoms and continuity if and only if there exists $\mu \in \Delta(S)$ with $\mu(A) \leq \frac{1}{2}$, $u : X \to \mathbb{R}$ continuous and strictly increasing and with $u(0) = 0$, $u(\bar{x}) = 1$, such that $\succcurlyeq$ is represented by*

$$V(f) = \int u(f(s))\mu(ds).$$

Moreover, $V'(f) = \int u'(f(s))\mu'(ds)$ represents $\succcurlyeq$ if and only if $\mu = \mu'$ and there exist $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$ such that $u' = au + b$

Proof. See the Appendix. □

3.3. A general axiomatization of second order expected utility

Second-Order Expected Utility (SOEU) (see Grant et al., 2009; Strzalecki, 2011) is a model that ranks each AA act f according to

$$V(f) = \int \phi(u(f(s)))\mu(ds).$$

Such a criterion is consistent with Ellsberg-type behavior and allows for sensitivity to the source of uncertainty. The function ϕ models attitudes toward uncertainty, while u models attitudes toward risk. A formulation of this criterion was first offered by Neilson (1993) (see also Neilson, 2010, Theorem 1 and Cerreia-Vioglio et al., 2012, Proposition 3). Grant et al. (2009) provide an axiomatization with ϕ concave. As in Neilson (2010), I consider a preference relation $\succsim$ over the set F_{AA} that satisfies Savage axioms for all acts and von Neumann–Morgenstern's axioms when restricted to constant acts.

P1 $\succsim$ is complete and transitive.

A2 For every constant acts $x, y, z \in X$ and $\alpha \in (0, 1]$

$$x \succsim y \iff \alpha x + (1-\alpha)z \succsim y + (1-\alpha)z.$$

A3 For every $f, g, h, \in F_{AA}$ the sets $\{\alpha \in [0, 1] | \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0, 1] | h \succsim \alpha f + (1-\alpha)g\}$ are closed.

P2 For every $f, g, h, h' \in F_{AA}$ and $E \in \Sigma$,

$$fEh \succsim gEh \implies fEh' \succsim gEh'.$$

P3 For any $A \in \Sigma$ non-null

$$x \succ y \iff xAf \succ yAg.$$

P4 For $A, B \in \Sigma$ and $x \succ y, x', y'$ such that $x \succ y$, $x' \succ y'$it holds

$$xAy \succsim xBy \implies x'Ay' \succsim x'By'.$$

P5 $\exists x, y \in X$ s.t. $x \succ y$.

P6 For every $f, g \in F_{AA}$ and $x \in X$ such that $g \succ f$, there exists a Σ-measurable partition $(A_i)_{i=1}^n$ of S such that for every $i = 1, \ldots, n$, $g \succ xAf$ and $xAg \succ f$.

Finally, I require a continuity condition similar to the one considered earlier.

Continuity If $(f_n(s))_n$ converges preference-wise to $f(s)$ for every $s \in S$ and for some $x, y \in X$ it holds that $x \succsim f_n(s) \succsim y$ for every $s \in S$, then f_n converges preference-wise to f.

Theorem 7. *Assume that Σ is countably separated. $\succsim$ satisfies P1–P6, A2–A3 and continuity if and only if there exists $u : X \to \mathbb{R}$ with $[0, 1] \subseteq u(X)$, $\phi : u(X) \to \mathbb{R}$ continuous and strictly increasing with $\phi(0) = 0$ and $\phi(1) = 1$ and $\mu \in \Delta_{na}(S)$ such that $\succsim$ is represented by $V : F_{AA} \to \mathbb{R}$ defined by*

$$V(f) = \int \phi(u(f(s)))\mu(ds).$$

Moreover, u is unique up to increasing affine transformations and for a given u, μ is unique and ϕ is unique up to increasing affine transformations over $u(X)$.

The proof I provide combines the AA approach with the techniques developed in this paper. In particular, usual arguments can be used to show that there exist $u : X \to \mathbb{R}$ and $I : B_0(\Sigma, u(X)) \to \mathbb{R}$ such that $\succsim$ is represented by

$$V(f) = I(u(f)).$$

Savage's axioms then imply that I satisfies all the conditions in Theorem 5 are satisfied, so that there exists $\phi : u(X) \to \mathbb{R}$ such that

$$V(f) = \int \phi(u(f(s)))\mu(ds).$$

Proof. See the Appendix. □

The non-atomicity of μ can be relaxed by imposing the unlikely atoms axiom as in Theorem 6.

4. Discussion

4.1. Extensions further applications

The findings from the previous section are built upon the assumptions of monetary consequences and countably additive probabilities. The former assumption can be relaxed by considering vector-valued acts. Proposition 3 in the Appendix discusses how to generalize Theorem 5 for this class of acts. The assumption of countably additive probabilities could be relaxed by using generalizations of the Radon–Nykodym theorem that hold for finitely additive measures (e.g., see Maynard, 1979).

Schmeidler (1986) provides an integral representation theorem (see also the earlier work by De Giorgi and Letta, 1977) in which standard additivity is relaxed to comonotonic additivity, i.e. additivity restricted to functions that are comonotonic. Two measurable functions f, g are comonotonic if

$$\left(f(\omega)-f\left(\omega'\right)\right)\left(g(\omega)-g\left(\omega'\right)\right)\geqslant 0,\quad \forall \omega,\omega'\in\Omega.$$

Comonotonic additivity neither implies nor is implied by disjoint additivity. A natural extension of the approach in this paper is to consider integral representation theorems that combine these two additivity conditions. For instance, one may want to restrict additivity to functions that have disjoint support and are comonotonic. It is natural to conjecture that such a condition, along with other appropriate ones, would lead to the representation in terms of the Choquet integral

$$I(f)=\int u(f(\omega))\nu(d\omega), \tag{5}$$

for every $f\in B(\mathcal{F},K)$, where $\nu:\mathcal{F}\to\mathbb{R}$ is a capacity and

$$\int u(f(\omega))\nu(d\omega)=\int_0^\infty \nu(u(f)\geq t)dt+\int_{-\infty}^0[\nu(u(f)\geq t)-\nu(S)]dt.$$

Gilboa (1987) provides an axiomatization of Choquet expected utility (CEU) in a Savage setting. A representation as in (5) would provide a way to axiomatize CEU with a different approach, possibly allowing for generalizations such as state-dependent utility.

A different setting in which these techniques can be applied is to recursive models under ambiguity (see Strzalecki, 2013). Here an act f denotes a plan, which is a collection of acts f_t representing state-contingent consumption at time t. Preferences conditional upon (t,ω) are represented by a functional $V_t(f,\omega)$ that satisfies the recursive relation

$$V_t(f,\omega)=u\left(f_t(\omega)\right)+\beta\phi^{-1}(\mathbb{E}_\mu\phi(V_{t+1}(f,\cdot))).$$

In particular, when $\phi(x)=-e^{-\frac{x}{\theta}}$ for $\theta>0$, this would lead to the multiplier preferences of Hansen and Sargent (2001). These and other generalizations are left to future research.

4.2. Connections with decision theory

As discussed earlier, the mathematical approach used in this paper is connected to the work of Wakker and Zank (1999), who also use the Radon–Nykodym theorem to derive an expected utility representation. Their result can be seen as a generalization of Theorem 2: by adding strict monotonicity and dropping strong scale-invariance, I admits the representation

$$I(f)=\int u(\omega,f(\omega))\mu(d\omega),$$

where for all $\omega\in\Omega$, $u(\omega,\cdot)$ is continuous and strictly increasing. Castagnoli and LiCalzi (2006) obtain an analogous representation and apply it to a procedure to rank real valued acts which they call benchmarking.

Kopylov (2010) refines Savage's theory when probabilities are countably additive. Differently from my approach, Kopylov's techniques are based on qualitative probability and in particular are related to his earlier paper Kopylov (2007). Theorems 2 and 5 borrow from his approach based countably separated σ-algebras.

Mackenzie (2019) generalizes Villegas's result to allow qualitative probabilities to be represented by probability measures that can have atoms. His paper provides two separate condition that relax non-atomicity: unlikely atoms and atom swarming. Because I restrict the attention to countably separated σ-algebras, the axiom I use in Theorem 6 is equivalent to his unlikely atoms axiom. However, the axiom I use is actually based on a condition developed by Martin and Mizel (1964), Theorem 1.8. Ha-Huy (2019) provides an axiomatization of expected utility based on the atom swarming axiom. A condition in the spirit of atom swarming also appears in Martin and Mizel (1964, p. 363, Theorem 1.11). Abdellaoui and Wakker (2019) substantially generalize Savage's theorem by replacing P6 with two axioms: archimedeanity and solvability. They obtain a subjective expected utility representation in which the probability satisfies a condition called range solvability.

As shown by Theorem 7, the results in this paper can be used outside of a Savage setting. In a recent working paper, Klibanoff et al. (2019) apply Theorem 5 to obtain an axiomatization of the so-called smooth ambiguity model (Klibanoff et al., 2005) under a symmetry assumption on preferences. Here the state space has the product structure $S=\Omega^\infty$, where Ω is a compact metric space. In an AA framework, they show that by imposing the Savage axioms only on acts that involve long-run frequency events preferences are represented by

$$U(f)=\int_{\Delta(\Omega)}\phi\left(\int u(f(s))\ell^\infty(ds)\right)\mu(d\ell),$$

where $\mu\in\Delta(\Omega)$ is a probability measures over marginal distributions (ℓ^∞ is the i.i.d. process with marginal ℓ), u is a von Neumann–Morgenstern utility function and ϕ is a continuous and strictly increasing function on the range of u.

Finally, this paper is related to the literature on axiomatizations of quasi-linear means developed in the late 1920s and the 1930s. The so-called Nagumo–Kolmogorov–De Finetti (see Nagumo, 1930; Kolmogorov, 1930; De Finetti, 1931, as well as Hardy et al., 1934) characterize quasi-linear means $M:D[a,b]\to\mathbb{R}$ defined by

$$M(F)=\psi^{-1}\left(\int\psi(x)F(dx)\right),$$

for every cumulative distribution function F on the interval $[a,b]$. See Muliere and Parmigiani (1993) and references contained therein for a discussion of the literature.

4.3. Connections with integral representation theory

Several papers in mathematics have developed similar representations to (1) and (2); see among many Martin and Mizel (1964), Friedman and Katz (1966a), Friedman and Katz (1966b), Mizel and Sundaresan (1968), Batt (1972), Palagallo (1976), Alò et al. (1977) and Hiai (1979). Other results of this type can be found in Buttazzo (1989) and Dal Maso (2012). These papers consider notions of additivity analogous to disjoint additivity. However, they differ from mine in that I consider functionals defined on the space of bounded and measurable functions, whereas this literature typically uses spaces of either integrable or continuous functions.

Appendix

Recall that an event $E \in \Sigma$ is non-null if there exist $x \in X$ and $f \in F$ such that $xEf \succ f$ (when considering Theorem 4, non-null means that $\bar{y}A0 \succ 0$).

Lemma 1. *Suppose that $\succcurlyeq$ over F satisfies P6. Then there exists a partition of S with three non-null events.*

Proof. This is a well-known fact; see for example (Fishburn, 1970, C9, p. 195). □

Lemma 2. *Suppose that $\succcurlyeq$ over F satisfies strong monotonicity and continuity. Then for every $f \in F$ there exists $x \in X$ such that $f \sim x$.*

Proof. For every $f \in F$, strong monotonicity implies that the sets $\{x \in X : x \succcurlyeq f\}$ and $\{x \in X : f \succcurlyeq x\}$ are non-empty. By continuity, they are both closed sets. Since X is connected, it follows that $\{x \in X : f \succcurlyeq x\} \cap \{x \in X : f \succcurlyeq x\} \neq \emptyset$. Hence there must be x such that $x \sim f$. □

The next lemma uses arguments from the proof of Proposition 3 in Wakker and Zank (1999).

Lemma 3. *Suppose that $\succcurlyeq$ over F satisfies P1, P2, continuity, the certainty equivalent axiom, there are at least three disjoint non-null events and there exist $\bar{k} \in X$ such that $\bar{k} > 0$ and $\bar{k} \succ 0$. Then there exists a continuous disjoint additive functional $V : F \to \mathbb{R}$ such that $V(F) = V(X)$, $V(\bar{k}) = 1$ that represents $\succcurlyeq$. Moreover, if W is another functional representing $\succcurlyeq$ with the same properties then $V = aW + b$ for scalars a, b with $a > 0$.*

Proof. Since there are at least three disjoint non-null events, any $f \in F$ can be written as

$$f = \sum_{i=1}^{N} x_i 1_{A_i}$$

where $\pi = \{A_1, \ldots, A_N\}$ is a partition of S with at least three non-null events. Consider $\succcurlyeq$ restricted to the set $X^\pi = \{\sum_{i=1}^{N} x_i 1_{A_i} : (x_i)_{i=1}^{N} \in X^N\}$. Since $\succcurlyeq$ satisfies P1, P2 and continuity, we can apply a well known result by Debreu (1959) and obtain continuous functions $V_{A_i}^\pi : X \to \mathbb{R}$ such that $\sum_{i=1}^{n} x_i 1_{A_i} \mapsto \sum_{i=1}^{N} V_{A_i}^\pi(x_i)$ represents $\succcurlyeq$ over X^π and satisfying $V_A^\pi(0) = 0$, $\sum_{i=1}^{N} V_{A_i}^\pi(\bar{k}) = 1$. Now for any two partitions π, π' of S with at least three non-null events, by looking at the common refinement of the two partitions the uniqueness result of Debreu's implies that the value of V_A^π is independent of the partition π. Hence, we can drop the superscript π in $V_{A_i}^\pi$.

Thus we can define the functional $V : F \to \mathbb{R}$ by

$$V(f) = \sum_{i=1}^{N} V_{A_i}(x_i).$$

V represents $\succcurlyeq$ on F. By the certainty equivalent axiom, $V(X) = V(F)$. Moreover, if W is another representation of $\succcurlyeq$ that satisfies $W(f) = \sum_{i=1}^{N} W_{A_i}(x_i)$ for every $f \in F$, then by standard arguments it follows that $W = \alpha V + \beta$ for some constants α, β with $\alpha > 0$.

Now I claim that V satisfies disjoint additivity. To see this, take $f, g \in F$ such that $\{s \in S : f(s) \neq 0\} \cap \{s \in S : g(s) \neq 0\} = \emptyset$. Then there exists a measurable partition $(A_{i=1})_{i=1}^{N+M}$ of S of non-null events and $N + M \geq 3$, such that

$$f = \sum_{i=1}^{N} x_i 1_{A_i},$$

and

$$g = \sum_{i=N+1}^{M} x_i 1_{A_i}.$$

Moreover, because f and g have disjoint support we can assume that

$$f + g = \sum_{i=1}^{N+M} x_i 1_{A_i},$$

so that

$$V(f+g) = \sum_{i=1}^{N+M} V_{A_i}(x_i) = \sum_{i=1}^{N} V_{A_i}(x_i) + \sum_{i=N+1}^{N+M} V_{A_i}(x_i) = V(f) + V(g),$$

as wanted. Finally, to see why V is continuous, let $f_n(s) \to f(s)$ for every $s \in S$ and suppose that $m \leq f_n \leq M$. By contradiction, assume that for a subsequence $(f_{n_j})_j$, there is $\varepsilon > 0$ such that $V(f_{n_j}) > V(f) + \varepsilon$ for every j. Because of the certainty equivalent axiom, we can find a constant act x with $V(f) < V(x) < V(f) + \varepsilon$. By continuity of $\succcurlyeq$ and since $x \succ f$, there exists a natural number k such that $x \succ f_{n_j}$ for all $j \geq k$, contradicting $V(f_{n_j}) > V(x)$ for all j. Hence, no such subsequence can exist. The same reasoning can be used to show that there can be no $\varepsilon > 0$ such that $V(f_{n_j}) < V(f) - \varepsilon$ for every j. Hence, $V(f_n)$ must converge to $V(f)$ as desired. □

A.1. Proof of Proposition 1

Consider μ' such that $\mu \ll \mu'$. Let $u'(\omega, x) = u(\omega, x)\frac{d\mu'}{d\mu}(\omega) \geq 0$. Then $I(f) = \int u(s, f(\omega))\mu(d\omega) = \int u(s, f(\omega))\frac{d\mu'}{d\mu}(\omega)\mu'(d\omega)$ for every $f \in B_0(\mathcal{F}, K)$.

Conversely, consider $u' : \Omega \times K \to \mathbb{R}$ measurable for every $x \in K$ and $\mu' \in \Delta(\Omega)$ such that $I(f) = \int u'(\omega, f(\omega))\mu'(d\omega)$. First note that $u'(\omega, x) \geq 0$ μ'-a.s. for every $x \in K$. This implies that $\mu \ll \mu'$: if $\mu'(A) = 0$, then $\mu(A) = I(\bar{k}1_A) = \int_A u'(\omega, k)\mu'(d\omega) = 0$. Finally, $\int u'(\omega, f(\omega))\mu'(d\omega) = \int u(\omega, f(\omega))\mu(d\omega) = \int u(\omega, f(\omega)) \frac{d\mu'}{d\mu}(\omega)\mu'(d\omega)$ for every $f \in B_0(\mathcal{F}, K)$. It follows that

$$\int_A (u(\omega, x)\frac{d\mu'}{d\mu}(\omega) - u'(\omega, x))\mu'(d\omega) = 0,$$

for every $A \in \mathcal{F}$ and $x \in K$ which implies $u(\omega, x)\frac{d\mu'}{d\mu}(\omega) = u'(\omega, x)$ μ'-a.s. for every $x \in K$ as desired.

A.2. Proof of Theorem 6

It is routine to check that the axioms are necessary.

For sufficiency, first note that P6 is satisfied on $S \setminus A$. More precisely, take $x \in X$ and $f, g \in F$ such that $f \succ g$. Let $\mathcal{A} \subseteq \mathcal{F}$ be the class of event that can be partitioned into events $A_1, \ldots, A_n$ such that $xA_if \succ g$ and $f \succ xA_ig$ for all $i = 1, \ldots, n$. Clearly if $B, A \in \mathcal{A}$ then $B \cup A \in \mathcal{A}$. Now I claim that $S \setminus A \in \mathcal{A}$. Suppose not. By applying Lemma 5 from Kopylov (2010), we get that there exists a decreasing sequence $(A_i)_{i=1}^{\infty}$ of subsets of $S \setminus A$ with $A_i \in \mathcal{A}$ for every i such that $\cap_{i=1}^{\infty} A_i$ is either empty or a singleton. Thus $g \succcurlyeq xA_if$ or $xA_ig \succcurlyeq f$ for all i. Note that xA_ig and xA_if converge pointwise to g and f, possibly with the exception of one point $s \in S \setminus A$, which is a null set since A_i is contained in $S \setminus A$ for every i. By continuity we get that $g \succcurlyeq f$, a contradiction. Thus by Lemma 1 there are at least three disjoint non-null events. By Lemma 3 we can construct a continuous, disjoint additive functional $V : F \to \mathbb{R}$ with $V(\bar{x}) = 1$ that represents $\succcurlyeq$. Moreover, strong monotonicity and P4 imply that V is strictly monotone and strongly scale-invariant. Using unlikely atoms and P4 with get that $\bar{x}S \setminus A0 \succcurlyeq \bar{x}A0$. It follows that

$$V(\bar{x}1_A) \leq V(\bar{x}1_{S\setminus A}) = 1 - V(\bar{x}1_A),$$

where the last equality follows by disjoint additivity. Hence we get $V(\bar{x}1_A) \leq \frac{1}{2}$, so that by applying Theorem 3 we obtain the desired result.

A.3. Proof of Theorem 7

It is routine to check that all the axioms are necessary.

For sufficiency, note that thanks to P1, A2 and A3 there exists $u : X \to \mathbb{R}$ affine and that represents $\succsim$ over constant acts (for example, see Cerreia-Vioglio et al., 2011a, Proposition 1). Moreover, u can be chosen so that $[0, 1] \subseteq u(X)$. Now define the preference $\succsim^*$ relation on $B_0(\Sigma, u(X))$ by

$$\xi \succsim^* \zeta \text{ if there exists } f, g \in F_{AA} \text{ such that } u(f) = \xi, u(g) = \zeta \text{ and } f \succsim g,$$

for every $\zeta, \xi \in B_0(\Sigma, u(X))$. It is routine to check that $\succsim^*$ is a well-defined weak order over $B_0(\Sigma, u(X))$. I claim that $\succsim^*$ satisfies all the axioms in Theorem 5. To see why P2 is satisfied, take ζ, ξ, ζ', ξ' and $A \in \Sigma$. Suppose that $\zeta A\zeta' \succsim^* \xi A\zeta'$. Then there exist f, g, h, h' such that $\zeta A\zeta' = u(f)Au(h)$, $\xi A\zeta' = u(g)Au(h)$, $\xi A\xi' = u(g)Au(h')$, $\zeta A\xi' = u(f)Au(h')$ and $fAh \succsim gAh$. Since $\succsim$ satisfies P2, it follows that $fAh' \succsim gAh'$ which implies $\zeta A\xi' \succsim^* \xi A\xi'$ as desired. Showing that $\succsim^*$ satisfies strong monotonicity, P4 and P6 are proved with the same strategy. To see why continuity is satisfied, suppose that $\zeta_n(s) \to \zeta(s)$ for every $s \in S$ and for some m, M $m \leq \zeta_n(s) \leq M$ for every $s \in S$. Take a sequence $(f_n)_n$ that satisfies $u(f_n) = \zeta_n$, so that $x \succsim f_n(s) \succsim y$ for some $x, y \in X$. Because $\zeta_n(s)$ converges to $\zeta(s)$, $f_n(s)$ converges preference-wise to $f(s)$ for every $s \in S$. Thus by continuity f_n converges preference-wise to f, which implies that ζ_n converges to ζ preference-wise as desired.

By Theorem 5 it follows that there exist $\mu \in \Delta_{na}(S)$ and $\phi : u(X) \to \mathbb{R}$ continuous and strictly increasing with $\phi(0), \phi(1) = 1$ such that

$$V(\zeta) = \int \phi(\zeta(s))\mu(ds) \quad \forall \zeta \in B_0(S, u(X)).$$

Hence

$$f \succsim g \iff u(f) \succsim^* u(g) \iff \int \phi(u(f(s)))\mu(ds) \geq \int \phi(u(g(s)))\mu(ds),$$

as desired.

Uniqueness of the representation follows by Theorem 5.

A.4. A generalization of Theorem 2 with vector-valued functions

Suppose that $K \subseteq V$, where V is a topological vector space. Here $B_0(\mathcal{F}, K)$ is the set of all the simple vector-valued functions with range contained in K. To extend Theorem 2 it is necessary to appropriately adapt the conditions on I to this more general setting. Generalize monotonicity in the following way: say that I is *monotone* if $I(f(\omega)) \geq I(g(\omega))$ for every $\omega \in \Omega$ implies $I(f) \geq I(g)$. To generalize the notion of a normalized functional, say that I is *normalized* if there exist $x, y \in K$ such that $I(x) = 1$ and $I(y1_A) = 0$ for every $A \in \mathcal{A}$. Say that I is strongly scale-invariant if for every $A, B \in \mathcal{F}$ and $z \in K$, $I(x1_A + y1_{A^c}) = I(x1_B + y1_{B^c}) \implies I(z1_A + y1_{A^c}) = I(z1_B + y1_{B^c})$. Continuity is extended in a natural way: say that I is *continuous* if for every sequence $(f_n)_n$ in $B_0(\mathcal{F}, K)$ such that for some constant m, M it holds that $m \leq I(f_n(\omega)) \leq M$ for every $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$ for every $\omega \in \Omega$ implies $I(f_n) \to I(f)$. Say that I is *non-atomic* if $I(z1_A + y1_{A^c}) = 0$ for every $z \in K$ and finite set A. Finally, disjoint additivity needs to be modified in the following way: for every $f, g \in B_0(\mathcal{F}, K)$ such that $\{\omega \in \Omega : g(\omega) \neq y\} \cap \{\omega \in \Omega : f(\omega) \neq y\} = \emptyset$ it holds that

$$I(f1_{\{\omega:f(\omega)\neq y\}} + g1_{\{\omega:g(\omega)\neq y\}} + y1_{\{\omega:f(\omega)=y,g(\omega)=y\}}) = I(g1_{\{\omega:g(\omega)\neq y\}} + y1_{\{\omega:g(\omega)=y\}}) + I(f1_{\{\omega:f(\omega)\neq y\}} + y1_{\{\omega:f(\omega)=y\}}).$$

Note that if y is the zero vector, then this notion reduces to the original notion of disjoint additivity.

Proposition 2. *Suppose that $\mathcal{F}$ is countably separated. $I : B_0(\mathcal{F}, K) \to \mathbb{R}$ is normalized, disjoint additive, monotone, strongly scale-invariant, continuous and non-atomic if and only if there exist $\mu \in \Delta_{na}(\Omega)$ and a continuous function $u : X \to \mathbb{R}$ with $u(x) = 1$, $u(y) = 0$ such that*

$$I(f) = \int u(f(\omega))\mu(d\omega).$$

Moreover, the representation is unique.

Proof. I briefly sketch that all the conditions are sufficient for the representation. Define $\mu(A) = I(x1_A + y1_{A^c})$ and $\phi_z(A) = I(z1_A + y1_{A^c})$. Observe that $\mu \in \Delta_{na}(\Omega)$. To see this, note that since $I(x) \geq I(x1_A(\omega) + y1_{A^c}(\omega)) \geq I(y)$ for every $\omega \in \Omega$, monotonicity implies that $1 = \mu(\Omega) = I(x) \geq I(x1_A + y1_{A^c}) \geq 0$. Disjoint additivity implies that $\mu(A \cup B) = I(x1_{A\cup B} + y1_{A^c\cap B^c}) = I(x1_A + x1_B + y1_{A^c\cap B^c}) = I(x1_A + y_{A^c}) + I(x1 + y1_{B^c}) = \mu(A) + \mu(B)$. Finally, non-atomicity implies that $\mu(\{\omega\}) = 0$ for every singleton $\{\omega\}$. By the same reasoning, ϕ_z is a signed measure for every $z \in K$. Moreover, by strong scale invariance $\phi_z \ll \mu$ for every z. The proof then follows the exact same step as the proof of Theorem 2. $\square$

Whenever K is a connected subset of V, the previous result can be used to extend Theorem 5 to these more general outcome spaces. Let the set of outcomes $X \subseteq V$ be a connected subset of V and consider the set of simple acts F that map states in S to consequences in X. Consider a relation $\succcurlyeq$ on F. The following axioms permit an extension of Theorem 5 to these general vector-valued acts. Observe that these are exactly Savage's axioms plus the additional continuity condition.

P1 $\succcurlyeq$ is complete and transitive.

P2 For every $f, g, h, h' \in F$ and $E \in \Sigma$,

$$fEh \succcurlyeq gEh \implies gEh' \succcurlyeq gEh'.$$

P3 For any $A \in \Sigma$ non-null

$$x \succ y \iff xAa \succcurlyeq yAb.$$

P4 For every $A, B \in \Sigma$ and $x, y, x', y' \in X$ such that $x \succ y$, $x' \succ y'$ it holds

$$xAy \succcurlyeq xBy \implies x'Ay' \succcurlyeq x'By'.$$

P5 There exist $x, y \in X$ such that $x \succ y$.

P6 For every $f, g \in F$ and $x \in X$ such that $f \succ g$, there exists a Σ-measurable partition $(A_i)_{i=1}^n$ of S such that for every $i = 1, \dots, n$, $g \succ x_{A_i}f$ and $x_{A_i}g \succ f$.

Continuity If $(f_n)_n$ is a sequence in F such that $f_n(\omega) \to f(s)$ for all $s \in S$ in the topology of V and for some $x', y' \in X$ it holds that $y' \succcurlyeq f_n(s) \succcurlyeq x'$ for all $s \in S$, then $(f_n)_n$ converges to f in preference.

Proposition 3. *Suppose that Σ is countably separated. $\succcurlyeq$ satisfies P1–P6 and continuity if and only if there exist $\mu \in \Delta_{na}(S)$, $u : X \to \mathbb{R}$ continuous and strictly increasing and with $u(x) = 1$, $u(y) = 0$, such that $\succcurlyeq$ is represented by*

$$V(f) = \int u(f(s))\mu(ds).$$

Moreover, $V'(f) = \int u'(f(s))\mu'(ds)$ represents $\succcurlyeq$ if and only if $\mu = \mu'$ and there exist $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$ such that $u' = au + b$

Proof. I briefly sketch the sufficiency of the axioms. Observe that Lemma 3 can be applied to $\succcurlyeq$ as long as X is connected (see Wakker, 1989 Wakker, Theorem III.6.6 for a generalization of Debreu's result to arbitrary connected topological spaces). P1, continuity and P3 imply that for every $f \in F$ there exists $x \in X$ such that $f \sim x$. Thus by Lemma 3 it possible to construct a continuous and disjoint additive functional that represents $\succcurlyeq$. The remaining axioms can be used to show that V satisfies all the condition in Proposition 2, which implies that V has the desired representation. □

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